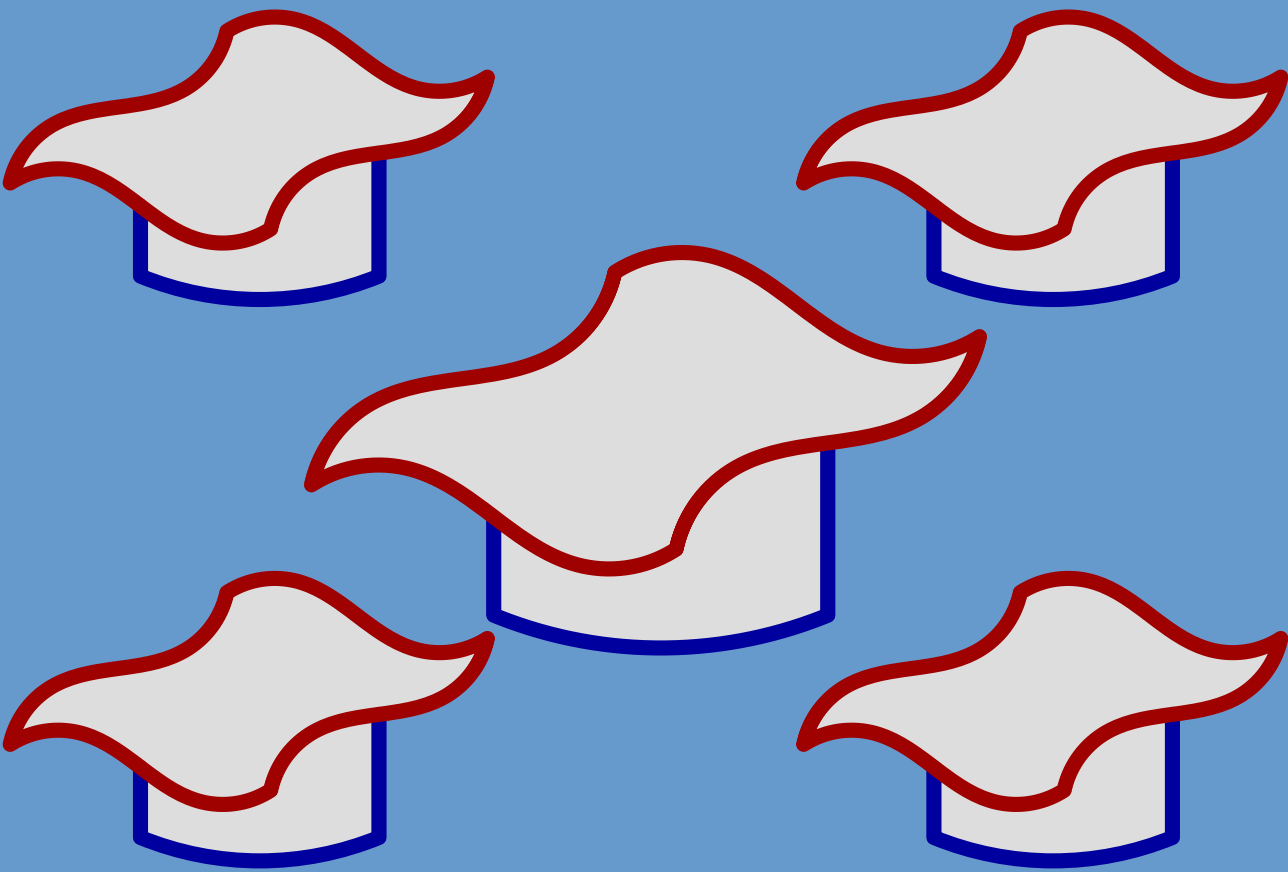
Tensor Analysis and Differential Geometry



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Chapter 1 Preface

This is a translation of the lecture notes of Jan de Graaf about Tensor Calculation and Differential Geometry. Originally the mathematical student W.A. van den Broek has written these lecture notes in 1995. During the years, several appendices have been added and all kind of typographical corrections have been done. Chapter 1 is rewritten by Jan de Graaf.
To make this translation is not asked to me. I hope that this will be, also for me, a good lesson in the Tensor Calculation and Differential Geometry. If you want to make comments, let me here, see the front page of this translation,

René van Hassel (started: April 2009)

In June 2010, the Mathematical Faculty asked me to translate the lecture notes of Jan de Graaf.
Pieces of text characterised by "RRvH:" are additions made by myself.
The text is typed in ConTeXt, a variant of TeX, see context.

Chapter 2 Multilinear Algebra

Section 2.1 Vector Spaces and Bases

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$.

Comment(s): 2.1.1

- For every basis $\{\mathbf{e}_i\}$ of V and for every vector $\mathbf{x} \in V$ there exists an unique ordered set of real numbers $\{x^i\}$ such that $\mathbf{x} = x^i \mathbf{e}_i$.

Definition 2.1.1 The numbers $\{x^i\}$ are called the contravariant components of the vector $\mathbf{x}$ with respect tot the basis $\{\mathbf{e}_i\}$. □

Convention(s): 2.1.1

- Contravariant components are organized in a $n \times 1$-matrix, also known as a columnvector. The columnvector belonging to the contravariant components x^i is notated as $\mathbf{X}$, so

$$\mathbf{X} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} = (x^1, \cdots, x^n)^T.$$

- The Vector Space of the real columnvectors of length n is notated by $\mathbb{R}^n$.

Definition 2.1.2 The amount of n columnvectors $\mathbf{E}_i$, defined by

$$\mathbf{E}_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T,$$

where there becomes a number 1 on the i-th position, is a basis of $\mathbb{R}^n$. This basis is called the standard basis of $\mathbb{R}^n$. □

Notice(s): 2.1.1

- $\mathbf{X} = x^i \mathbf{E}_i$.
- To every basis $\{\mathbf{e}_i\}$ of V can be defined a bijective linear transformation $\mathcal{E} : V \rightarrow \mathbb{R}^n$ by $\mathcal{E}\mathbf{x} = \mathbf{X}$. Particularly holds that $\mathcal{E}\mathbf{e}_i = \mathbf{E}_i$. A bijective linear transformation is also called a isomorphism. By choosing a basis $\{\mathbf{e}_i\}$ of V and by defining the correspondent isomorphism $\mathcal{E}$, the Vector Space V is "mapped". With the help of $\mathbb{R}^n$, V is provided with a "web of coordinates".

Comment(s): 2.1.2

- For every pair of bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_{i'}\}$ of V, there exist an unique pair of ordered real numbers $A^i_{i'}$ and $A^{i'}_i$ such that $\mathbf{e}_i = A^{i'}_i \mathbf{e}_{i'}$ and $\mathbf{e}_{i'} = A^i_{i'} \mathbf{e}_i$.

Convention(s): 2.1.2

- The numbers $A^i_{i'}$ are organized in a $n \times n$-matrix, which is notated by $\mathbf{A}_,$. So $\mathbf{A}_, = [A^i_{i'}]$ with i the row index and i' the column index. The matrix $\mathbf{A}_,$ is the change-of-coordinates matrix from the basis $\{\mathbf{e}_i\}$ to the basis $\{\mathbf{e}_{i'}\}$.
- The numbers $A^{i'}_i$ are also organized in a $n \times n$-matrix, which is notated by $\mathbf{A}'$. So $\mathbf{A}' = [A^{i'}_i]$ with i' the row index and i the column index. The matrix $\mathbf{A}'$ is the change-of-coordinates matrix from the basis $\{\mathbf{e}_{i'}\}$ to the basis $\{\mathbf{e}_i\}$.
- The contravariant components of the vector $\mathbf{x}$ with respect to the basis $\{\mathbf{e}_{i'}\}$ are notated by $x^{i'}$ and the belonging columnvector by $\mathbf{X}'$.

Notice(s): 2.1.2

- On the one hand holds $\mathbf{e}_i = A^{i'}_i \mathbf{e}_{i'} = A^{i'}_i A^j_{i'} \mathbf{e}_j$ and on the other hand $\mathbf{e}_i = \delta^i_j \mathbf{e}_j$, from what follows that $A^{i'}_i A^j_{i'} = \delta^i_j$. On the same manner is to deduce that $A^i_{i'} A^{j'}_i = \delta^{i'}_{j'}$. The δ^i_j and $\delta^{i'}_{j'}$ are the Kronecker delta's. Construct with them the identity matrices $\mathbf{I} = [\delta^i_j]$ and $\mathbf{I}^{'}_{,} = [\delta^{i'}_{j'}]$, then holds $\mathbf{A}^{'}\mathbf{A}_{,} = \mathbf{I}^{'}_{,}$ and $\mathbf{A}_{,}\mathbf{A}^{'} = \mathbf{I}$. Evidently is that $(\mathbf{A}_{,})^{-1} = \mathbf{A}^{'}$ and $(\mathbf{A}^{'})^{-1} = \mathbf{A}_{,}$.
- Carefulness is needed by the conversion of an index notation to a matrix notation. In an index notation the order does not play any rule, on the other hand in a matrix notation is the order of crucial importance.
- For some vector $\mathbf{x} \in V$ holds on one hand $\mathbf{x} = x^i \mathbf{e}_i = x^i A^{i'}_i \mathbf{e}_{i'}$ and on the other hand $\mathbf{x} = x^{i'} \mathbf{e}_{i'}$, from which follows that $x^{i'} = x^i A^{i'}_i$. This expresses the relationship between the contravariant components of a vector $\mathbf{x}$ with respect to the basis $\{\mathbf{e}_i\}$ and the contravariant components with respect to the basis $\{\mathbf{e}_{i'}\}$. Analoguous is seen that $x^i = x^{i'} A^i_{i'}$. In the matrix notation this correspondence is written as $\mathbf{X} = \mathbf{A}_{,}\mathbf{X}^{'}$ and $\mathbf{X}^{'} = \mathbf{A}^{'}\mathbf{X}$.
- Putting the expression $x^{i'} = x^i A^{i'}_i$ and $\mathbf{e}_{i'} = A^i_{i,} \mathbf{e}_i$ side by side, then is seen that the coordinates x^i "transform" with the inverse $A^{i'}_i$ of the change-of-coordinates matrix $A^i_{i'}$. That is the reason of the strange 19th century term <u>*contravariant*</u> components.
- Out of the relation $x^{i'} = x^i A^{i'}_i$ follows that $\dfrac{\partial x^{i'}}{\partial x^i} = A^{i'}_i$.

Notation(s):

- The basisvector $\mathbf{e}_i$ is also written as $\dfrac{\partial}{\partial x^i}$ and the basisvector $\mathbf{e}_{i'}$ as $\dfrac{\partial}{\partial x^{i'}}$. At this stage this is pure formal and there is not attached any special meaning to it. Look to the formal similarity with the chain rule. If the function f is enough differentiable then

$$\frac{\partial f}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial f}{\partial x^{i'}} = A^{i'}_i \frac{\partial f}{\partial x^{i'}},$$

so $\frac{\partial}{\partial x^i} = A^{i'}_i \frac{\partial}{\partial x^{i'}}$, which corresponds nicely with $\mathbf{e}_i = A^{i'}_i \mathbf{e}_{i'}$.

Example(s): 2.1.1 Let $S = \{\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1)\}$ be the standard basis of $\mathbb{R}^2$ and let $T = \{\mathbf{e}_{1'} = \frac{1}{\sqrt{5}}(1, -2)^T, \mathbf{e}_{2'} = \frac{1}{\sqrt{5}}(2, 1)^T\}$ be an orthonormal basis of $\mathbb{R}^2$.
The coordinates of $\mathbf{e}_{1'}$ and $\mathbf{e}_{2'}$ are given with respect to the standard basis S.
The matrix

$$A_{,} = \begin{pmatrix} \mathbf{e}_{1'} & \mathbf{e}_{2'} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$

$A_{,}((\mathbf{e}_{1'})_T) = (\mathbf{e}_{1'})_S$ and $A_{,}((\mathbf{e}_{2'})_T) = (\mathbf{e}_{2'})_S$.
The matrix $A_{,}$ is chosen orthogonal, such that $A^{'} = (A_{,})^{-1} = (A_{,})^T$ and

$$\mathbf{x}_T = A^{'}\mathbf{x}_S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{8}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

the result is that

$$\mathbf{x} = \frac{8}{\sqrt{5}}\mathbf{e}_1 + \frac{-1}{\sqrt{5}}\mathbf{e}_2 = 2\mathbf{e}_{1'} + 3\mathbf{e}_{2'}.$$

□

Section 2.2 The Dual Space. The concept dual basis

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$.

Definition 2.2.1 A linear function $\widehat{\mathbf{f}}$ on V is a transformation of V to $\mathbb{R}$, which satisfies the following propertie

$$\widehat{\mathbf{f}}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\,\widehat{\mathbf{f}}(\mathbf{x}) + \beta\,\widehat{\mathbf{f}}(\mathbf{y}),$$

for all $\mathbf{x}, \mathbf{y} \in V$ and every $\alpha, \beta \in \mathbb{R}$. □

Definition 2.2.2 The Dual Space V^* belonging to the Vector Space V is the set of all linear functions on V, equipt with the addition and the scalar multiplication. For every pair linear functions $\widehat{\mathbf{f}}$ and $\widehat{\mathbf{g}}$ on V, the function $(\widehat{\mathbf{f}}+\widehat{\mathbf{g}})$ is defined by $(\widehat{\mathbf{f}}+\widehat{\mathbf{g}})(\mathbf{x}) = \widehat{\mathbf{f}}(\mathbf{x}) + \widehat{\mathbf{g}}(\mathbf{x})$. For every linear function $\widehat{\mathbf{f}}$ and every real number α, the linear function $(\alpha\,\widehat{\mathbf{f}})$ is defined by $(\alpha\,\widehat{\mathbf{f}})(\mathbf{x}) = \alpha\,(\widehat{\mathbf{f}}(\mathbf{x}))$. It is easy to control that V^* is a Vector Space. The linear functions $\widehat{\mathbf{f}} \in V^*$ are called covectors or covariant 1-tensors. □

Definition 2.2.3 To every basis $\{\mathbf{e}_i\}$ of V and for every covector $\widehat{\mathbf{f}} \in V^*$ is defined an ordered set of real numbers $\{f_i\}$ by $f_i = \widehat{\mathbf{f}}(\mathbf{e}_i)$. These numbers f_i are called the covariant components of the covector $\widehat{\mathbf{f}}$ with respect to the basis $\{\mathbf{e}_i\}$. □

Convention(s): 2.2.1

- The covariant components of a covector are organized in a $1 \times n$-matrix, also known as a rowvector. The rowvector belonging to the covariant components f_i is notated as $\widehat{\mathbf{F}}$, so

$$\widehat{\mathbf{F}} = (f_1, f_2, \cdots, f_n)$$

- The Vector Space of the real rowvectors of length n is notated by $\mathbb{R}_n$.

Definition 2.2.4 The amount of n rowvectors $\widehat{\mathbf{E}}^{\,i}$, defined by

$$\widehat{\mathbf{E}}^{\,i} = (0, \cdots, 0, 1, 0, \cdots, 0),$$

where there becomes a number 1 on the i-th position, is a basis of $\mathbb{R}_n$. This basis is called the standard basis of $\mathbb{R}_n$. □

Notice(s): 2.2.1

- $\widehat{\mathbf{F}} = f_i \widehat{\mathbf{E}}^i$.
- $\widehat{\mathbf{f}}(\mathbf{x}) = \widehat{\mathbf{F}}\mathbf{X}$, where $\widehat{\mathbf{F}}$ represents the rowvector of the covariant components of the covector $\widehat{\mathbf{f}}$.
- To every basis $\{\mathbf{e}_i\}$ of V belong n linear functions $\widehat{\mathbf{e}}^k$, defined by $\widehat{\mathbf{e}}^k(\mathbf{x}) = x^k = (\mathbf{E}_k)^T \mathcal{E}\mathbf{x}$. Keep in mind, that every $\widehat{\mathbf{e}}^k$ most of the time is determined by the entire basis $\{\mathbf{e}_i\}$.

Lemma 2.2.1 The collection $\{\widehat{\mathbf{e}}^i\}$ forms a basis of V^*. □

Proof Let $\widehat{\mathbf{g}} \in V^*$. For every $\mathbf{x} \in V$ holds that

$$\widehat{\mathbf{g}}(\mathbf{x}) = \widehat{\mathbf{g}}(x^i \mathbf{e}_i) = x^i \widehat{\mathbf{g}}(\mathbf{e}_i) = g_i x^i = g_i(\widehat{\mathbf{e}}^i(\mathbf{x})) = (g_i \widehat{\mathbf{e}}^i)(\mathbf{x}),$$

so $\widehat{\mathbf{g}} = g_i \widehat{\mathbf{e}}^i$. The Dual Space V^* is spanned by the collection $\{\widehat{\mathbf{e}}^i\}$. The only thing to prove is that $\{\widehat{\mathbf{e}}^i\}$ are linear independent. Assume that $\{\alpha_i\}$ is a collection of numbers such that $\alpha_i \widehat{\mathbf{e}}^i = 0$. For every j holds that $\alpha_i \widehat{\mathbf{e}}^i(\mathbf{e}_j) = \alpha_i \delta^i_j = \alpha_j = 0$. Hereby is proved that $\{\widehat{\mathbf{e}}^i\}$ is a basis of V^*. □

Consequence(s):

- The Dual Space V^* of V is, just as V itself, n-dimensional.

Definition 2.2.5 The basis $\{\widehat{\mathbf{e}}^i\}$ is called the to $\{\mathbf{e}_i\}$ belonging <u>dual basis</u> of V^*. □

Notice(s): 2.2.2

- To every choice of a basis $\{\mathbf{e}_i\}$ in V there exists a bijection $\mathcal{E} : V \to \mathbb{R}^n$, see Notice(s) 2.1.1. A choice of a basis in V leads to a dual basis in V^*. Define a linear transformation $\mathcal{E}^* : \mathbb{R}_n V \to V^*$ by $\mathcal{E}^* \widehat{\mathbf{E}}^i = \mathbf{e}_i$. This linear transformation is bijective. There holds that $\mathcal{E}^* \widehat{\mathbf{F}} = \widehat{\mathbf{f}}$ and also that

$$\widehat{\mathbf{f}}(\mathbf{x}) = \mathcal{E}^* \widehat{\mathbf{F}}(\mathbf{x}) = \widehat{\mathbf{F}}(\mathcal{E}\mathbf{x}) = \widehat{\mathbf{F}}\mathbf{X}$$

Lemma 2.2.2 Let $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_{i'}\}$ be bases of V and consider their belonging dual bases $\{\widehat{\mathbf{e}}^i\}$ and $\{\widehat{\mathbf{e}}^{i'}\}$.

- The transition between the basis $\{\widehat{\mathbf{e}}^i\}$ and the basis $\{\widehat{\mathbf{e}}^{i'}\}$ is given by the known matrices A' and $A_,$, see Comment(s) 2.1.2. It goes on the following way:

$$\{\widehat{\mathbf{e}}_i\} = A^i_{i'}\{\widehat{\mathbf{e}}_{i'}\} \text{ and } \{\widehat{\mathbf{e}}_{i'}\} = A^{i'}_i\{\widehat{\mathbf{e}}_i\}.$$

□

Proof The transition matrices between the bases $\{\widehat{\mathbf{e}}^i\}$ and $\{\widehat{\mathbf{e}}^{i'}\}$ are notated by $[B^{i'}_i]$ and $[B^i_{i'}]$. On one side holds $\widehat{\mathbf{e}}^{i'}(\mathbf{e}_j) = \delta^i_j$ and on the other hand holds

$$\widehat{\mathbf{e}}^{i'}(\mathbf{e}_j) = (B^i_{i'}\widehat{\mathbf{e}}^{i'})(A^{j'}_j\mathbf{e}_{j'}) = B^i_{i'}A^{j'}_j\delta^{i'}_{j'} = B^i_{i'}A^{i'}_j,$$

such that $B^i_{i'}A^{i'}_j = \delta^i_j$. It is obvious that $B^i_{i'} = A^i_{i'}$. □

Notice(s): 2.2.3

- Changing of a basis means that the components of $\widehat{\mathbf{y}}$ transform as follows

$$\widehat{\mathbf{y}} = y_{i'}\widehat{\mathbf{e}}^{i'} = y_{i'}A^{i'}_i\widehat{\mathbf{e}}^i = (A^{i'}_i y_{i'})\widehat{\mathbf{e}}^i = y_i\widehat{\mathbf{e}}^i = y_iA^i_{j'}\widehat{\mathbf{e}}^{j'} = y_iA^i_{j'}\widehat{\mathbf{e}}^{i'}.$$

  In matrix notation

$$\widehat{\mathbf{Y}}_, = \widehat{\mathbf{Y}}\mathbf{A}_, \qquad \widehat{\mathbf{Y}} = \widehat{\mathbf{Y}}_,\mathbf{A}' \quad (= \widehat{\mathbf{Y}}_,(\mathbf{A}_,)^{-1})$$

- Putting the expression $y_{i'} = A^i_{i'}y_i$ and $\mathbf{e}_{i'} = A^i_{i,}\mathbf{e}_i$ side by side, then is seen that the coordinates y_i "transform" just as the basis vectors. That is the reason of the strange 19th century term <u>*covariant*</u> components.

Notation(s):

- The basisvector $\widehat{\mathbf{e}}^i$ is also written as $d\,x^i$ and the dual basisvector $\widehat{\mathbf{e}}^{i'}$ as $d\,x^{i'}$. At this stage this is pure formal and there is not attached any special meaning to it. Sometimes is spoken about "infinitesimal growing", look to the formal similarity

$$d\,x^{i'} = \frac{\partial x^{i'}}{\partial x^i}d\,x^i = A^{i'}_i\,d\,x^{i'}.$$

  This corresponds nicely with $\widehat{\mathbf{e}}^{i'} = A^{i'}_i\widehat{\mathbf{e}}^i$.

Section 2.3 The Kronecker tensor

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$ and it's Dual Space V^*.

Notation(s):

- The covector $\widehat{\mathbf{f}} : \mathbf{x} \longmapsto \widehat{\mathbf{f}}(\mathbf{x})$ will hence'forth be written as $\widehat{\mathbf{f}} : \mathbf{x} \longmapsto < \widehat{\mathbf{f}}, \mathbf{x} >$.
- Sometimes is written $\widehat{\mathbf{f}} = < \widehat{\mathbf{f}}, \cdot >$. The "argument" $\mathbf{x}$ leaves "blanc".

Definition 2.3.1 The function of 2 vectorvariables $< \cdot, \cdot >: V^* \times V \to \mathbb{R}$ is called the Kronecker tensor. □

Notice(s): 2.3.1

- Covectors can only be filled in at the first entrance of $< \cdot, \cdot >$, so elements out of V^*. At the second entrance there can be filled in vectors, so elements out of V. The Kronecker tensor is <u>not</u> an "inner product", because every entrance can only receive only its own type of vector.
- The Kronecker tensor is a <u>linear functions in every separate variable</u>. That means that

$$\forall_{\widehat{\mathbf{u}}, \widehat{\mathbf{v}} \in V^*} \forall_{\mathbf{z} \in V} \forall_{\alpha, \beta \in \mathbb{R}} : < \alpha \widehat{\mathbf{u}} + \beta \widehat{\mathbf{v}}, \mathbf{z} > = \alpha < \widehat{\mathbf{u}}, \mathbf{z} > + \beta < \widehat{\mathbf{v}}, \mathbf{z} >, \quad \text{en}$$

$$\forall_{\widehat{\mathbf{u}} \in V^*} \forall_{\mathbf{x}, \mathbf{y} \in V} \forall_{\gamma, \delta \in \mathbb{R}} : < \widehat{\mathbf{u}}, \gamma \mathbf{x} + \delta \mathbf{y} > = \gamma < \widehat{\mathbf{u}}, \mathbf{x} > + \delta < \widehat{\mathbf{u}}, \mathbf{y} > .$$

- The pairing between the basisvectors and the dual basisvectors provides:

$$< \widehat{\mathbf{e}}^i, \mathbf{e}_j > = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \qquad ,$$

  the famous "Kronecker delta".
- To every fixed chosed $\mathbf{a} \in V$, there can be looked to the linear function $\widehat{\mathbf{u}} : \mathbf{x} \longmapsto < \widehat{\mathbf{u}}, \mathbf{a} >$ on V^*! This linear function belongs to the dual of the Dual Space of V, notation: $(V^*)^* = V^{**}$. A co-covector, so to speak. The following lemma shows that in the finite dimensional case, V^{**} can be indentified with V, without "introducing extra structures". The proof needs some skills with the abstract linear algebra.

Lemma 2.3.1 Bidual Space

Let $\widehat{\widehat{\mathbf{f}}} : V^* \to \mathbb{R}$ be a linear function. Then there exists an unique vector, say $\mathbf{a} \in V$, such that for all $\widehat{\mathbf{u}} \in V^*$ holds that: $\widehat{\widehat{\mathbf{f}}}(\widehat{\mathbf{u}}) = < \widehat{\mathbf{u}}, \mathbf{a} >$.
(Without any risk there can be posed that $V^{**} = V$). □

Proof Choose a vector $\mathbf{a} \in V$. Define the "evaluation function" $\widehat{\widehat{\delta}}_{\mathbf{a}} : V^* \to \mathbb{R}$ by

$$\widehat{\widehat{\delta}}_{\mathbf{a}}(\widehat{\mathbf{u}}) = < \widehat{\mathbf{u}}, \mathbf{a} > .$$

Look at the linear transformation $J : V \to V^{**}$, defined by $J\mathbf{x} = \widehat{\widehat{\delta}}_{\mathbf{x}}$. The linear transformation J is injective. If $(J\mathbf{x})(\widehat{\mathbf{u}}) = < \widehat{\mathbf{u}}, \mathbf{x} > = 0$ for all $\widehat{\mathbf{u}} \in V^*$, then $\mathbf{x}$ has to be $\mathbf{0}$. Take for $\widehat{\mathbf{u}}$ successively the elements of a "dual basis". Because furthermore $\dim V^{**} = \dim V^* = \dim V = n < \infty$, J has also to be surjective. This last is justified by the dimension theorem. The bijection $J : V \to V^{**}$ "identifies" V^{**} and V without making extra assumptions. □

Comment(s): 2.3.1

- If the vector $\mathbf{x} \in V$ is meant as a linear function on V^*, there is written $\mathbf{x} = < \cdot, \mathbf{x} >$. A vector perceived as linear function is called a <u>contravariant 1-tensor</u>.
- To emphasize that the covector $\widehat{\mathbf{y}}$ is a linear function, it is written as $\widehat{\mathbf{y}} = < \widehat{\mathbf{y}}, \cdot >$. Such a covector is called a <u>covariant 1-tensor</u>.
- The question announces if V and V^* can be identified. They have the same dimension. The answer will be <u>that this can not be done</u> without extra assumptions. Later on will be showed, that the choice of an inner product will determine how V and V^* are identified to each other.

Section 2.4 Linear Transformations. Index-gymnastics

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$ and it's Dual Space V^*.
- A linear transformation $\mathcal{R}$ from V to V.
- A linear transformation $\mathcal{P}$ from V^* to V^*.
- A linear transformation $\mathcal{G}$ from V to V^*.
- A linear transformation $\mathcal{H}$ from V^* to V.

Notation(s):

- Let $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_{i'}\}$ be bases of V, then is written

$$\begin{aligned} \mathcal{R}\mathbf{e}_i &= R^j_i \mathbf{e}_j\,, \\ \mathcal{R}\mathbf{e}_{i'} &= R^{j'}_{i'} \mathbf{e}_{j'}\,. \end{aligned} \tag{2.1}$$

  This means that the contravariant components of the vector $\mathcal{R}\mathbf{e}_i$ with respect to the basis $\{\mathbf{e}_j\}$ are notated by R^j_i and the contravariant components of the vector $\mathcal{R}\mathbf{e}_{i'}$ with respect to the basis $\{\mathbf{e}_{j'}\}$ are notated by $R^{j'}_{i'}$.
- These two unique labeled collection of numbers are organised in $n \times n$-matrices, notated by respectively $\mathcal{R}$ and $\mathcal{R}^{'}_{,}$. So $\mathcal{R} = [R^j_i]$ and $\mathcal{R}^{'}_{,} = [R^{j'}_{i'}]$. Hereby are j and j' the rowindices and i and i' are the columnindices.

Notice(s): 2.4.1

- With the help of the transisition matrices $A_{,}$ and $A^{'}$ there can be made a link between the matrices $\mathcal{R}$ and $\mathcal{R}^{'}_{,}$. There holds namely

$$\mathcal{R}\mathbf{e}_{i'} = \mathcal{R}(A^i_{i'}\mathbf{e}_i) = A^i_{i'}\mathcal{R}\mathbf{e}_i = A^i_{i'}R^j_i\mathbf{e}_j = A^i_{i'}R^j_iA^{j'}_j\,\mathbf{e}_{j'}.$$

  Compare this relation with (2.1) and it is easily seen that $R^{j'}_{i'} = A^i_{i'}R^j_iA^{j'}_j$. In matrix notation, this relation is written as $\mathcal{R}^{'}_{,} = A^{'}\mathcal{R}A_{,}$. Also is to deduce that $R^j_i = A^{i'}_iR^{j'}_{i'}A^j_{j'}$, which in matrix notation is written as $\mathcal{R} = A_{,}\mathcal{R}^{'}_{,}A^{'}$.
- The relations between the matrices, which represent the linear transformation $\mathcal{R}$ with respect to the bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_{i'}\}$, are now easily to deduce. There holds namely

$$\mathcal{R}^{'}_{,} = (A_{,})^{-1}\mathcal{R}A_{,} \quad \text{and} \quad \mathcal{R} = (A^{'})^{-1}\mathcal{R}^{'}_{,}A^{'}.$$

The other linear types of linear transformations can also be treated on almost the same way. The results are collected in figure 2.1.

Comment(s): 2.4.1 Comments to figure 2.1.

- $\forall_{\mathbf{x} \in V} \forall_{\widehat{\mathbf{y}} \in V^*} :< \widehat{\mathbf{y}}, \mathcal{R}\mathbf{x} > = < \mathcal{P}\widehat{\mathbf{y}}, \mathbf{x} >$ holds exactly if $P = R$, so if $P^i_j = R^i_j$ holds exactly.
- $\forall_{\mathbf{x} \in V} \forall_{\mathbf{z} \in V} :< \mathcal{G}\mathbf{y}, \mathbf{z} > = < \mathcal{G}\mathbf{z}, \mathbf{x} >$ holds exactly if $G^T = G$, so if $g_{ji} = g_{ij}$ holds exactly. In such a case the linear transformation $\mathcal{G} : V \rightarrow V^*$ is called symmetric.
- Some of the linear transformations in the table can be composed by other linear transformations out of the table. If $\mathcal{R} = \mathcal{H} \circ \mathcal{G}$ then is obvious $R^k_j x^j \mathbf{e}_k = \mathcal{R}\mathbf{x} = (\mathcal{H} \circ \mathcal{G})\mathbf{x} = h^{kl} g_{jl} x^j \mathbf{e}_k$. In matrix notation: $RX = H(X^T G)^T = HG^T X$.
- If $\mathcal{P} = \mathcal{G} \circ \mathcal{H}$ then is obvious $P^k_j y_k \widehat{\mathbf{e}}^j = \mathcal{P}\mathbf{y} = \mathcal{G} \circ \mathcal{H}\widehat{\mathbf{y}} = h^{kl} g_{lj} y_k \widehat{\mathbf{e}}^j$. In matrix notation: $\widehat{Y}P = (H\widehat{Y}^T)^T G = \widehat{Y} H^T G$.

Coordinate free notation	Indexnotation = Componentnotation	Matrixnotation
No basis used **Order is fixed**	**Basis dependent** **Order is of no importance**	**Basis dependent** **Order is fixed**
Space		
$\mathbf{x} \in V$	$x \quad x^{i'} \quad x^{i''}$, with $\mathbf{x} = x^i \mathbf{e}_i = x^{i'} \mathbf{e}_{i'}$, etc. $x^i = < \widehat{\mathbf{e}}^i, \mathbf{x} > \quad x^{i'} = < \widehat{\mathbf{e}}^{i'}, \mathbf{x} >$ $x^{i'} = A^{i'}_{i} x^i \quad x^{i''} = A^{i''}_{i} x^i$, etc.	$X \quad X'$ $X' = A' X_{,}$ etc.
$\widehat{\mathbf{y}} \in V^*$	$y \quad y_{i'} \quad y_{i''}$, with $\widehat{\mathbf{y}} = y_i \widehat{\mathbf{e}}^i = y_{i'} \widehat{\mathbf{e}}^{i'}$, etc. $y_i = < \widehat{\mathbf{y}}, \mathbf{e}_i > \quad y_{i'} = < \widehat{\mathbf{y}}, \mathbf{e}_{i'} >$ $y_{i'} = A^{i}_{i'} y_i \quad y_{i''} = A^{i}_{i''} y_i$, etc.	$\widehat{Y} \quad \widehat{Y}_{,}$ $\widehat{Y} = \widehat{Y}_{,} A'$ etc.
$< \widehat{\mathbf{y}}, \mathbf{x} > \in \mathbb{R}$	$y_i x^i = y_{i'} x^{i'} \in \mathbb{R}$	$\widehat{Y} X = \widehat{Y}_{,} X' \in \mathbb{R}$
$\mathcal{R}$		
$\mathcal{R} : V \to V$	$\mathcal{R} \mathbf{e}_j = R^{i}_{j} \mathbf{e}_i \quad \mathcal{R} \mathbf{e}_{j'} = R^{i'}_{j'} \mathbf{e}_{i'}$ $R^{i}_{j} = < \widehat{\mathbf{e}}^i, \mathcal{R} \mathbf{e}_j >$ $R^{i'}_{j'} = < \widehat{\mathbf{e}}^{i'}, \mathcal{R} \mathbf{e}_{j'} > = A^{i'}_{i} A^{j}_{j'} x^j$	$R = [R^{i}_{j}]$, j is column index $R = A_{,} R'_{,} A'$
$\mathcal{R} \mathbf{x} \in V$	$R^{i}_{j} x^j = < \widehat{\mathbf{e}}^i, \mathcal{R} \mathbf{x} >$ $R^{i'}_{j'} x^{j'} = < \widehat{\mathbf{e}}^{i'}, \mathcal{R} \mathbf{x} > = A^{i'}_{i} R^{j}_{j} x^j$	column$(R^{i}_{j} x^j) = RX$ column$(R^{i'}_{j'} x^j) = R'_{,} X'$
$< \widehat{\mathbf{y}}, \mathcal{R} \mathbf{x} > \in \mathbb{R}$	$y_i R^{i}_{j} x^j = R^{i'}_{j'} x^{j'} y_{i'} \in \mathbb{R}$	$\widehat{Y} R X = \widehat{Y}_{,} R'_{,} X' \in \mathbb{R}$
$\mathcal{P}$		
$\mathcal{P} : V^* \to V^*$	$\mathcal{P} \widehat{\mathbf{e}}^i = P^{i}_{j} \widehat{\mathbf{e}}^j \quad \mathcal{P} \widehat{\mathbf{e}}^{i'} = P^{i'}_{j'} \widehat{\mathbf{e}}^{j'}$ $P^{i}_{j} = < \mathcal{P} \widehat{\mathbf{e}}^i, \mathbf{e}_j >$ $P^{i'}_{j'} = < \mathcal{P} \widehat{\mathbf{e}}^{i'}, \mathbf{e}_{j'} > = A^{i'}_{i} A^{j}_{j'} P^{i}_{j}$	$P = [P^{i}_{j}]$, j is column index $P'_{,} = A' P A_{,}$
$\mathcal{P} \widehat{\mathbf{y}} \in V^*$	$P^{j}_{i} y_j = < \mathcal{P} \widehat{\mathbf{y}}, \widehat{\mathbf{e}}_i >$ $P^{j'}_{i'} y_{j'} = < \mathcal{P} \widehat{\mathbf{y}}, \widehat{\mathbf{e}}_{i'} > = A^{i}_{i'} P^{j}_{i} y_j$	row$(P^{j}_{i} y_j) = \widehat{Y} P$ row$(P^{j'}_{i'} y_{j'}) = \widehat{\mathbf{Y}}_{,} P'_{,} = \widehat{\mathbf{Y}} P A$
$< \mathcal{P} \widehat{\mathbf{y}}, \mathbf{x} > \in \mathbb{R}$	$y_j P^{j}_{i} x^i = P^{j'}_{i'} y_{j'} x^{i'} \in \mathbb{R}$	$\widehat{Y} P X = \widehat{Y}_{,} P'_{,} X' \in \mathbb{R}$
$\mathcal{G}$		
$\mathcal{G} : V \to V^*$	$\mathcal{G} \mathbf{e}_i = g_{ij} \widehat{\mathbf{e}}^j \quad \mathcal{G} \mathbf{e}_{i'} = g_{i'j'} \widehat{\mathbf{e}}^{j'}$ $g_{ij} = < \mathcal{G} \mathbf{e}_i, \mathbf{e}_j >$ $g_{i'j'} = < \mathcal{G} \mathbf{e}_{i'}, \mathbf{e}_{j'} > = A^{i}_{i'} A^{j}_{j'} g_{ij}$	$G = [g_{ij}]$, j is column index $G_{,,} = [g_{i'j'}] = (A_{,})^T G A_{,}$
$\mathcal{G} \mathbf{x} \in V^*$	$g_{ij} x^i = < \mathcal{G} \mathbf{x}, \mathbf{e}_j >$ $g_{i'j'} x^{i'} = < \mathcal{G} \mathbf{x}, \mathbf{e}_{j'} > = A^{j}_{j'} g_{ij} x^i$	row$(g_{ij} x^i) = X^T G$ row$(g_{i'j'} x^{i'}) = (X')^T G_{,,}$ $= (X')^T (A_{,})^T G A_{,} = X^T G A_{,}$
$< \mathcal{G} \mathbf{x}, \mathbf{z} > \in \mathbb{R}$	$g_{ij} x^i z^j = g_{i'j'} x^{i'} z^{j'} \in \mathbb{R}$	$X^T G Z = (X')^T G_{,,} Z' \in \mathbb{R}$
$\mathcal{H}$		
$\mathcal{H} : V^* \to V$	$\mathcal{H} \widehat{\mathbf{e}}^k = h^{kl} \mathbf{e}_l \quad \mathcal{H} \widehat{\mathbf{e}}^{k'} = h^{k'l'} \mathbf{e}_{l'}$ $h^{kl} = < \widehat{\mathbf{e}}^k, \mathcal{H} \widehat{\mathbf{e}}^l >$ $h^{k'l'} = < \widehat{\mathbf{e}}^{k'}, \mathcal{H} \widehat{\mathbf{e}}^{l'} > = A^{k'}_{k} A^{l'}_{l} h^{kl}$	$H = [h^{kl}]$, l is column index $H'' = [h^{k'l'}] = A' H (A')^T$
$\mathcal{H} \widehat{\mathbf{y}} \in V^*$	$h^{kl} y_l = < \widehat{\mathbf{e}}^k, \mathcal{H} \widehat{\mathbf{y}} >$ $h^{k'l'} y_{l'} = < \widehat{\mathbf{e}}^{k'}, \mathcal{H} \widehat{\mathbf{y}} > = A^{k'}_{k} h^{kl} y_l$	column$(h^{kl} y_l) = H \widehat{Y}^T$ column$(h^{k'l'} y_{l'}) = H'' (\widehat{\mathbf{Y}}_{,})^T$ $= A' H (\widehat{\mathbf{Y}}_{,} A')^T$
$< \widehat{\mathbf{u}}, \mathcal{H} \widehat{\mathbf{y}} > \in \mathbb{R}$	$h^{kl} u_k y_l = h^{k'l'} u_{k'} y_{l'} \in \mathbb{R}$	$\widehat{U} H \widehat{Y}^T = \widehat{U}_{,} H'' (\widehat{Y}_{,})^T \in \mathbb{R}$

Figure 2.1 Indexgymnastic.

Section 2.5 Inner product

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$ and it's Dual Space V^*.
- Bases $\{\mathbf{e}_i\}$,$\{\mathbf{e}_{i'}\}$ in V.
- The corresponding dual bases $\{\widehat{\mathbf{e}}^i\}$,$\{\widehat{\mathbf{e}}^{i'}\}$ in V^*.

Definition 2.5.1 An Euclidean inner product on V is a transformation from $V \times V$ to $\mathbb{R}$, which satisfies the following properties

i. $\forall_{\mathbf{x},\mathbf{y} \in V}: \quad (\mathbf{x},\mathbf{y}) = (\mathbf{y},\mathbf{x})$

ii. $\forall_{\mathbf{x},\mathbf{y},\mathbf{z} \in V} \forall_{\alpha,\beta \in \mathbb{R}}: \quad (\alpha\mathbf{x} + \beta\mathbf{y},\mathbf{z}) = \alpha(\mathbf{x},\mathbf{z}) + \beta(\mathbf{y},\mathbf{z})$

iii. $\forall_{\mathbf{x} \in V}: \quad \mathbf{x} \neq \mathbf{0} \Leftrightarrow (\mathbf{x},\mathbf{x}) > 0$

□

Notice(s): 2.5.1 In the mathematics and physics there are often used other inner products. They differ from the Euclidean inner product and it are variations on the conditions of Definition 2.5.1 i and Definition 2.5.1 iii. Other possibilities are

a. instead of Def. 2.5.1 i:
$\forall_{\mathbf{x},\mathbf{y} \in V}: \quad (\mathbf{x},\mathbf{y}) = -(\mathbf{y},\mathbf{x})$

b. instead of Def. 2.5.1 iii:
$\forall_{\mathbf{x} \in V,\, \mathbf{x} \neq \mathbf{0}} \exists_{\mathbf{y} \in V}: \quad (\mathbf{x},\mathbf{y}) \neq 0$

Clarification(s): 2.5.1

- Condition Def. 2.5.1 iii implies condition Ntc. 2.5.1 b. Condition Ntc. 2.5.1 b is weaker then condition Def. 2.5.1 iii.
- In the theory of relativity, the Lorentz inner product plays some rule and it satisfies the conditions Def. 2.5.1 i, Def. 2.5.1 ii and Ntc. 2.5.1 b.
- In the Hamiltonian mechanics an inner product is defined by the combination of the conditions Ntc. 2.5.1 a, Def. 2.5.1 ii and Ntc. 2.5.1 b. The Vector Space V is called a symplectic Vector Space. There holds that $\dim V$ = even. ('Phase space')
- If the inner product satisfies condition Def. 2.5.1 i, the inner product is called symmetric. If the inner product satisfies condition Ntc. 2.5.1 a then the inner product is called antisymmetric.

Definition 2.5.2 let $\{\mathbf{e}_i\}$ be a basis of V and define the numbers $g_{ij} = (\mathbf{e}_i, \mathbf{e}_j)$. The matrix $G = [g_{ij}]$, where i is the row index and j is the column index, is called the Gram matrix. □

Notation(s):

- If the inverse G^{-1} of the Gram matrix exists then it is notated by $G^{-1} = [g^{kl}]$, with k the row index and j the column index. Notate that $g_{ik}g^{kj} = \delta_i^j$ and $g^{li}g_{ik} = \delta_k^l$.

Theorem 2.5.1 Consider an inner product $(\cdot, \cdot)$ on V which satisfies the conditions: Def. 2.5.1 i or Ntc. 2.5.1 a, Def. 2.5.1 ii and Ntc. 2.5.1 b.

a. There exists a bijective linear transformation $\mathcal{G} : V \to V^*$ such that

$$\forall_{\mathbf{x},\mathbf{y} \in V} : (\mathbf{x}, \mathbf{y}) = < \mathcal{G}\mathbf{x}, \mathbf{y} > \text{ and } \forall_{\widehat{\mathbf{z}} \in V^*}, \forall_{\mathbf{y} \in V} : < \widehat{\mathbf{z}}, \mathbf{y} > = (\mathcal{G}^{-1}\widehat{\mathbf{z}}, \mathbf{y}).$$

b. The Gram matrix is invertible.
c. There holds $\mathcal{G}\mathbf{e}_i = g_{ik}\widehat{\mathbf{e}}^k$. If $\mathbf{x} = x^i\mathbf{e}_i$, then is $\mathcal{G}\mathbf{x} = x^i g_{ik}\widehat{\mathbf{e}}^k$.
d. There holds $\mathcal{G}^{-1}\widehat{\mathbf{e}}^l = g^{li}\mathbf{e}_i$. If $\widehat{\mathbf{y}} = y_l\widehat{\mathbf{e}}^l$, then is $\mathcal{G}^{-1}\mathbf{y} = y_l g^{li}\mathbf{e}_i$.

□

Proof

a. Take a fixed $\mathbf{u} \in V$ and define the linear function $x \mapsto (\mathbf{u}, \mathbf{x})$. Then there exists a $\widehat{\mathbf{u}} \in V^*$ such that for all $\mathbf{x} \in V$ holds: $< \widehat{\mathbf{u}}, \mathbf{x} > = (\mathbf{u}, \mathbf{x})$. The addition $\mathbf{u} \mapsto \widehat{\mathbf{u}}$ seems to be a linear transformation. This linear transformation is called $\mathcal{G} : V \to V^*$. So $\widehat{\mathbf{u}} = \mathcal{G}\mathbf{u}$.
Because $\dim(V) = \dim(V^*) < \infty$ the bijectivity of $\mathcal{G}$ is proved by proving that $\mathcal{G}$ is injective. Assume that there exists some $\mathbf{v} \in V$, $\mathbf{v} \neq \mathbf{0}$ such that $\mathcal{G}\mathbf{v} = 0$. Then holds for all $\mathbf{x} \in V$ that $0 = < \mathcal{G}\mathbf{v}, \mathbf{x} > = (\mathbf{v}, \mathbf{x})$ and this is in contradiction with Ntc. 2.5.1 b.
b. G is invertiblle if and only if G^T is invertible. Assume that there is a columnvector $X \in \mathbb{R}^n$, $X \neq O$ such that $G^T X = O$. Then the rowvector $X^T G = O$. With $\mathbf{x} = \mathcal{E}^{-1}X \neq O$ follows that the covector $\mathcal{E}^*(X^T G) = \mathcal{G}\mathbf{x} = \mathbf{0}$. This is contradiction with the bijectivity of $\mathcal{G}$.
c. The components of $\mathcal{G}\mathbf{e}_i$ are calculated by $< \mathcal{G}\mathbf{e}_i, \mathbf{e}_k > = (\mathbf{e}_i, \mathbf{e}_k) = g_{ik}$. There follows that $\mathcal{G}\mathbf{e}_i = g_{ik}\widehat{\mathbf{e}}^k$ and also that $\mathcal{G}\mathbf{x} = \mathcal{G}(x^i\mathbf{e}_i) = x^i g_{ik}\widehat{\mathbf{e}}^k$.
d. Out of $\mathcal{G}(g^{li}\mathbf{e}_i) = g^{li}\mathcal{G}\mathbf{e}_i = g^{li}g_{ik}\widehat{\mathbf{e}}^k = \delta_k^l\widehat{\mathbf{e}}^k = \widehat{\mathbf{e}}^l$ follows that $\mathcal{G}^{-1}\widehat{\mathbf{e}}^l = g^{li}\mathbf{e}_i$. At last $\mathcal{G}^{-1}\widehat{\mathbf{y}} = \mathcal{G}^{-1}(y_l\widehat{\mathbf{e}}^l) = y_l g^{li}\mathbf{e}_i$. □

Comment(s): 2.5.1

- The second part of Theorem 2.5.1 a gives the <u>Representation Theorem of Riesz</u>: For every linear function $\widehat{g} \in V^*$ there exists exactly one vector $g \in V$, namely $g = \mathcal{G}^{-1}\widehat{g}$ such that $\widehat{g}(\widehat{x}) = (\widehat{g}, \widehat{x})$ for every $x \in V$.
- If the inner product satisfies condition Def. 2.5.1 i, then the Gram matrix is symmetric, i.e. $G^T = G$. If the inner product satisfies condition Ntc. 2.5.1 a then the Gram matrix is antisymmetric, i.e. $G^T = -G$.
- For every $\mathbf{x}, \mathbf{y} \in V$ holds $(\mathbf{x}, \mathbf{y}) = (x^i \mathbf{e}_i, y^j \mathbf{e}_j) = x^i y^j g_{ij} = X^T G Y$. Note that in the symmetric case, $g_{ij} = g_{ji}$, such that $(\mathbf{x}, \mathbf{y}) = Y^T G X$.
- If there are pointed two bases in V, $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_{i'}\}$ then holds

  $$g_{i'j'} = (\mathbf{e}_{i'}, \mathbf{e}_{j'}) = (A^i_{i'} \mathbf{e}_i, A^j_{j'} \mathbf{e}_j) = A^i_{i'} A^j_{j'} (\mathbf{e}_i, \mathbf{e}_j) = A^i_{i'} A^j_{j'} g_{ij}.$$

  The numbers $g_{i'j'}$ are put in a matrix with the name $G_{,,}$. So $G_{,,} = [g_{i'j'}]$ with j' the column index and i' the row index, such that in matrix notation can be written $G_{,,} = (A_,)^T G A_,$.

Starting Point(s): Concerning Inner Products:

- In the follow up, so also in the next paragraphs, the inner product is assumed to satisfy the conditions Def. 2.5.1 i, Def. 2.5.1 ii and Ntc. 2.5.1 b, unless otherwise specified. So the Gram matrix will always be symmetric.

Definition 2.5.3 In the case of an Euclidean inner product the length of a vector $\mathbf{x}$ is notated by $| \mathbf{x} |$ and is defined by

$$| \mathbf{x} | = \sqrt{(\mathbf{x}, \mathbf{x})}.$$

□

Lemma 2.5.1 In the case of an Euclidean inner product holds for every pair of vectors $\mathbf{x}$ and $\mathbf{y}$

$$| (\mathbf{x}, \mathbf{y}) | \leq | \mathbf{x} | | \mathbf{y} | .$$

This inequality is called the <u>inequality of Cauchy-Schwarz</u>. □

Definition 2.5.4 In the case of an Euclidean inner product the angle ϕ between the vectors $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$ is defined by

$$\varphi = \arccos\left(\frac{(\mathbf{x},\mathbf{y})}{|\,\mathbf{x}\,||\,\mathbf{y}\,|}\right).$$

□

Section 2.6 Reciproke basis

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$.
- An inner product $(\cdot,\cdot)$ on V.
- Bases $\{\mathbf{e}_i\}$,$\{\mathbf{e}_{i'}\}$ in V.
- The corresponding dual bases $\{\widehat{\mathbf{e}}^i\}$,$\{\widehat{\mathbf{e}}^{i'}\}$ in V^*.

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$.
- An inner product $(\cdot,\cdot)$ on V.
- Bases $\{\mathbf{e}_i\}$,$\{\mathbf{e}_{i'}\}$ in V.
- The corresponding dual bases $\{\widehat{\mathbf{e}}^i\}$,$\{\widehat{\mathbf{e}}^{i'}\}$ in V^*.

Definition 2.6.1 To the basis $\{\mathbf{e}_i\}$ in V is defined a second basis $\{\mathbf{e}^i\}$ in V which is defined by $\{\mathbf{e}^i\} = \mathcal{G}^{-1}\widehat{\mathbf{e}}^i = g^{ij}\mathbf{e}_j$. This second basis is called the to the first basis belonging <u>reciproke basis</u>. □

Comment(s): 2.6.1

- Out of $\mathbf{e}^i = g^{ij}\mathbf{e}_j$ follows that $g_{ki}\mathbf{e}^i = g_{ki}g^{ij}\mathbf{e}_j = \delta^j_k\mathbf{e}_j = \mathbf{e}_k$. These relations express the relation between a basis and its belonging reciproke basis. It is important to accentuate that the reciproke basis depends on the chosen inner product on V. If there was chosen another inner product on V then there was attached another reciproke basis to the same basis.
- The to the reciproke basis belonging Gram matrix is easily to calculate

$$(\mathbf{e}^i, \mathbf{e}^j) = g^{ik}g^{jl}(\mathbf{e}_k, \mathbf{e}_l) = g^{ik}g^{jl}g_{kl} = \delta^i_l g^{jl} = g^{ji}.$$

- There holds that $(\mathbf{e}^i, \mathbf{e}_j) = g^{il}(\mathbf{e}^l, \mathbf{e}^j) = g^{il}g_{lj} = \delta^i_j$. In such a case is said that the vectors $\mathbf{e}^i$ and $\mathbf{e}_j$ for every $i \neq j$ are staying perpendicular to each other.

Lemma 2.6.1 Let $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_{i'}\}$ be bases of V and consider there belonging reciproke bases $\{\mathbf{e}^i\}$ and $\{\mathbf{e}^{i'}\}$. The transistion matrix from the basis $\{\mathbf{e}_i\}$ to the basis $\{\mathbf{e}_{i'}\}$ is given by $A^{'}$ and the transistion matrix the other way around is given by $A_{,}$. So $\mathbf{e}^i = A^i_{i'}\mathbf{e}^{i'}$ and $\mathbf{e}^{i'} = A^{i'}_i\mathbf{e}^i$. □

Proof It follows direcly out of the transistions between dual bases, see Lemma 2.2.2. The proof can be repeated but then without "dual activity".
Notate the transition matrices between the bases $\{\mathbf{e}^i\}$ and $\{\mathbf{e}^{i'}\}$ by $B_{,} = [B^{i'}_i]$ and $B^{'} = [B^i_{i'}]$, so $\mathbf{e}^i = B^i_{i'}\mathbf{e}^{i'}$ and $\mathbf{e}^{i'} = B^{i'}_i\mathbf{e}^i$. On one hand holds $(\mathbf{e}^i, \mathbf{e}_j) = \delta^i_j$ and on the other hand $(\mathbf{e}^i, \mathbf{e}_j) = (B^i_{i'}\mathbf{e}^{i'}, A^{j'}_j\mathbf{e}_{j'}) = B^i_{i'}A^{j'}_j\delta^{i'}_{j'} = B^i_{i'}A^{j'}_j$, so $B^i_{i'}A^{j'}_j = \delta^i_j$. Obviously are $B_{,}$ and $A^{'}$ each inverse. The inverse of $A^{'}$ is given by $A_{,}$, so $B_{,} = A_{,}$. Completely analogous is to deduce that $B^{'} = A^{'}$. □

Definition 2.6.2 The numbers (factorisation coefficients) x_i in the representation of $\mathbf{x} = x_i\mathbf{e}^i$ are called the <u>covariant components</u> of the vector $\mathbf{x}$ with respect to the basis $\{\mathbf{e}_i\}$. □

Comment(s): 2.6.2

- For the covariant components holds $x_i = x_j \delta_i^j = x_j(\mathbf{e}^j, \mathbf{e}_i) = (x_j \mathbf{e}^j, \mathbf{e}_i) = (\mathbf{x}, \mathbf{e}_i)$.
- For the contravariant components holds $x^i = x^j \delta_j^i = x^j(\mathbf{e}^i, \mathbf{e}_j) = (\mathbf{e}^i, x^j \mathbf{e}_j) = (\mathbf{e}^i, \mathbf{x})$.
- With respect to a second basis $\{\mathbf{e}_{i'}\}$ holds $x_{i'} = (\mathbf{x}, \mathbf{e}_{i'}) = (\mathbf{x}, A^i_{i'} \mathbf{e}_i) = A^i_{i'}(\mathbf{x}, \mathbf{e}_i) = A^i_{i'} x_i$.
- The covariant components transform on the same way as the basisvectors, this in contrast to the contravariant components. This explains the words covariant and contravariant. The scheme beneath gives some clarification.

$$\mathbf{e}_{i'} = A^i_{i'} \mathbf{e}_i \text{ (with: } A_{,}) \Rightarrow \begin{cases} x_{i'} = A^i_{i'} x_i & \text{covariant case with } A_{,} \\ \\ x^{i'} = A^{i'}_i x^i & \text{contravariant case with } A^{'} = (A_{,})^{-1} \end{cases}$$

- The mutual correspondence between the covariant and the contravariant components is described with the help of the Gram matrix and its inverse. There holds that $x_i = (\mathbf{x}, \mathbf{e}_i) = x^j(\mathbf{e}_j, \mathbf{e}_i) = g_{ji} x^j$ and for the opposite direction holds $x^i = (\mathbf{x}, \mathbf{e}^i) = x_j(\mathbf{e}^j, \mathbf{e}^i) = g^{ji} x_j$. With the help of the Gram matrix and its inverse the indices can be shifted "up" and "down".
- The inner product between two vectors $\mathbf{x}$ and $\mathbf{y}$ can be written on several manners

$$(\mathbf{x}, \mathbf{y}) = \begin{cases} x^i y^j g_{ij} & = x^i y_i \\ \\ x_i y_j g^{ij} & = x_i y^i. \end{cases}$$

Summarized:

$$\begin{array}{lcr}
x_{i'} = x_i A^i_{i'} & \Leftrightarrow & \widehat{\mathbf{X}}_{,} = \widehat{\mathbf{X}} A_{,} \\
x_i = g_{ij} x^j & \Leftrightarrow & \widehat{\mathbf{X}} = (G\mathbf{X})^T \\
(\mathbf{x}, \mathbf{y}) = x_i y^i & \Leftrightarrow & (\mathbf{x}, \mathbf{y}) = \widehat{\mathbf{X}} \mathbf{Y} \\
(\mathbf{x}, \mathbf{y}) = x_{i'} y^{i'} & \Leftrightarrow & (\mathbf{x}, \mathbf{y}) = \widehat{\mathbf{X}}_{,} \mathbf{Y}^{'} \\
(\mathbf{x}, \mathbf{y}) = g_{ij} x^i y^j & \Leftrightarrow & (\mathbf{x}, \mathbf{y}) = \mathbf{X}^T G \mathbf{Y} \\
(\mathbf{x}, \mathbf{y}) = g_{i'j'} x^{i'} y^{j'} & \Leftrightarrow & (\mathbf{x}, \mathbf{y}) = \left(\mathbf{X}^{'}\right)^T G_{,,} \mathbf{Y}^{'}
\end{array}$$

Conclusion(s): IMPORTANT:
To a **FIXED CHOSEN** inner product $(\cdot.\cdot)$ the concept of "dual space" can be ignored without any problems. **EVERY** preliminary formula with hooks "$< \cdot, \cdot >$" in it, gives a correct expression if the hooks "$< \cdot, \cdot >$" are replaced by "$(\cdot, \cdot)$" and if the caps " $\widehat{\cdot}$ " are kept away. There can be calculated on the customary way such as is done with inner products.

Section 2.7 Special Bases and Transformation Groups

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$.
- An inner product $(\cdot, \cdot)$ on V.

Lemma 2.7.1 For every invertible symmetric $n \times n$-matrix Q there exists a whole number $p \in \{0, \cdots, n\}$ and an invertible matrix A such that $A^T Q A = \Delta$, with $\Delta = \text{diag}(1, \cdots, 1, -1, \cdots, -1)$. The matrix Δ contains p times the number 1 and $(n - p)$ times th number -1. Notate $A = [A^j_i]$, $Q = [Q_{ij}]$ and $\Delta = [\Delta_{ij}]$, then holds $A^k_i Q_{kl} A^l_j = \Delta_{ij}$ in index notation. □

Proof Because of the fact that Q is symmetric, there exists an orthogonal matrix F such that

$$F^T Q F = \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n).$$

The eigenvalues of Q are ordered such that $\lambda_1 \geq \cdots \geq \lambda_n$. The eigenvalues $\lambda_i \neq 0$, because the matrix Q in invertible. Define the matrix

$$\mid \Lambda \mid^{-\frac{1}{2}} = \text{diag}(\mid \lambda_1 \mid^{-\frac{1}{2}}, \cdots, \mid \lambda_n \mid^{-\frac{1}{2}})$$

and next the matrix $A = F \mid \Lambda \mid^{-\frac{1}{2}}$, then holds

$$A^T Q A = \left(F \mid \Lambda \mid^{-\frac{1}{2}}\right)^T Q F \mid \Lambda \mid^{-\frac{1}{2}} = \mid \Lambda \mid^{-\frac{1}{2}} F^T Q F \mid \Lambda \mid^{-\frac{1}{2}} = \mid \Lambda \mid^{-\frac{1}{2}} \Lambda \mid \Lambda \mid^{-\frac{1}{2}} = \Delta,$$

with Δ the searched matrix. The number of positive eigenvalues of the matrix Q gives the nummer p. □

Theorem 2.7.1 **(Signature of an inner product)** There exists a whole number $p \in \{0, \cdots, n\}$ and a basis $\{\mathbf{e}_i\}$ of V such that

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} \text{ for } 1 \leq i \leq p \leq n \text{ and } 1 \leq j \leq n,$$
$$(\mathbf{e}_i, \mathbf{e}_j) = -\delta_{ij} \text{for } p < i \leq n \text{ and } 1 \leq j \leq n.$$

□

Proof Let $\{\mathbf{c}_i\}$ be a basis of V. Let Q be the corresponding Gram matrix, so $Q_{ij} = (\mathbf{c}_i, \mathbf{c}_j)$. This matrix is symmetrix and invertible. Lemma 2.7.1 gives that there exists an invertible matrix A such that $A^T Q A = \Delta = \mathrm{diag}(1, \cdots, 1, -1, \cdots, -1)$. Write $A = [A^i_{i'}]$ and define the set $\{\mathbf{c}_{i'}\}$ with $\mathbf{c}_{i'} = A^i_{i'} \mathbf{e}_i$. Since A is invertible, the set $\{\mathbf{c}_{i'}\}$ is a basis of V and there holda that

$$(\mathbf{c}_{i'}, \mathbf{c}_{j'}) = A^i_{i'} A^j_{j'} (\mathbf{c}_i, \mathbf{c}_j) = A^i_{i'} Q_{ij} A^j_{j'} = \Delta_{i'j'}.$$

Define $\mathbf{e}_i = \mathbf{c}_i$ and the searched basis is found. □

Comment(s): 2.7.1

- The previous proof shows that p is determined by the amount of positive eigenvalues of the Gram matrix of an arbitrary basis. This amount is for every basis the same, such that p is uniquely coupled tot the inner product on V. The number p is called the <u>signature</u> of that inner product. If for instance $p = 1$ then sometimes the signature is also notated by $(+, -, -, \cdots, -)$.

Definition 2.7.1 The to Theorem 2.7.1 belonging basis $\{\mathbf{e}_i\}$ is called an <u>orthonormal basis</u> of the Vector Space V. □

Notice(s): 2.7.1

- The Gram matrix belonging to an orthonormal basis is a diagonal matrix, with the first p diagonal elements equal to 1 and the remaining diagonal elements equal to -1. There holds the following relationship between the reciproke basis of an orthonormal basis and the orthonormal basis itself,

$$\mathbf{e}^i = \mathbf{e}_i \text{ for } 1 \leq i \leq p \text{ and } \mathbf{e}^i = -\mathbf{e}_i \text{ for } p < i \leq n. \tag{2.2}$$

Definition 2.7.2 The set of transition matrices between orthonormal bases is called the transformation group. For every $p \in \{0, 1, \cdots, n\}$ and $q = n - p$ this transformation group is defined by

$$\mathcal{O}(p,q) = \{A \in \mathbb{R}^{n \times n} \mid A^T \Delta A = \Delta\}.$$

A special subgroup of it is

$$\mathcal{SO}(p,q) = \{A \in \mathcal{O}(p,q) \mid \det A = 1\}.$$

□

Example(s): 2.7.1

- If the inner product on V has signature $p = n$ then the group $\mathcal{O}(p,q)$ is exactly equal to the set of orthogonal matrices. This group is called the orthogonal group and is notated by $\mathcal{O}(n)$. An element out of the subgroup $\mathcal{SO}(n)$ transforms an orthogonal basis to an orthogonal basis with the same "orientation". Remember that the orthogonal matrices with determinent equal to 1 describe rotations around the origin.
- Let the dimension of V be equal to 4 and the inner product on V has signature $p = 1$. Such an inner product space is called Minkowski Space. The belonging group $\mathcal{O}(1,3)$ is called the Lorentz group and elements out of this group are called Lorentz transformations. Examples of Lorentz transformations are

$$A_1 = \begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with φ an arbitrary real number and

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & T & \\ 0 & & & \end{pmatrix},$$

where T an arbitrary element out of the orthogonal group $\mathcal{O}(3)$. □

Section 2.8 Tensors

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$.
- An inner product $(\cdot,\cdot)$ on V.

2.8.1 General Definition

Definition 2.8.1 The $\binom{r}{s}$-tensor on V, with $r = 0, 1, 2, \cdots$, $s = 0, 1, 2, \cdots$ is a function

$$T : \underbrace{V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times \cdots \times V}_{s \text{ times}} \to \mathbb{R},$$

$$\underbrace{\widehat{\mathbf{u}}; \widehat{\mathbf{v}}; \cdots; \widehat{\mathbf{z}}}_{r \text{ covectors } \subseteq V^*}; \underbrace{\mathbf{v}; \mathbf{w}; \cdots; \mathbf{y}}_{s \text{ vectors } \subseteq V} \mapsto \mathbb{R},$$

which is linear in each argument. This means that for every $\alpha, \beta \in \mathbb{R}$ and each "slot" holds that by way of example

$$T(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \cdots, \alpha\widehat{\mathbf{z}}_1 + \beta\widehat{\mathbf{z}}_2, \mathbf{v}; \mathbf{w}; \cdots; \mathbf{y}) =$$
$$\alpha T(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \cdots, \widehat{\mathbf{z}}_1, \mathbf{v}; \mathbf{w}; \cdots; \mathbf{y}) + \beta T(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \cdots, \widehat{\mathbf{z}}_2, \mathbf{v}; \mathbf{w}; \cdots; \mathbf{y}).$$

For more specification there is said that T is contravariant of order r and is covariant of order s. If holds that $p = r + s$ then there is sometimes spoken about a p-tensor. □

Comment(s): 2.8.1

- The <u>order</u> of the covectors $\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \cdots, \widehat{\mathbf{z}}$ and the order ofthe vectors $\mathbf{v}; \mathbf{w}; \cdots; \mathbf{y}$ is of importance! Most of the time, the value of T changes if the order of two covectors or vectors are changed. If a vector is changed with a covector the result is a meaningless expression.
- Sometimes a notation is used such that the covectors and the vectors are not splitted up into two separated groups, but are placed on an agreed fixed position. The order remains of importance and the previous remark is still valid!

2.8.2 $\binom{0}{0}$-tensor = scalar = number

If $p = 0$ there are no vectors or covectors to fill in. The following definition is just a convention.

Definition 2.8.2 A $\binom{0}{0}$-tensor is an alternative name for a real number. A $\binom{0}{0}$-tensor is also called a scalar. □

2.8.3 $\binom{1}{0}$-tensor = contravariant 1-tensor = vector

Definition 2.8.3 A $\binom{1}{0}$-tensor is a linear transformation of V^* to $\mathbb{R}$. □

Notice(s): 2.8.1

- Write the tensor as $\widehat{\mathbf{y}} \mapsto T(\widehat{\mathbf{y}})$. In accordeance with Lemma 2.3.1 there is some vector $\mathbf{a} \in V$ such that $T(\widehat{\mathbf{y}}) = < \widehat{\mathbf{y}}, \mathbf{a} >$ for all $\widehat{\mathbf{y}} \in V^*$. The set of $\binom{1}{0}$-tensors is exactly equal to the Vector Space V, the startingpoint.
- For every basis $\{\mathbf{e}_i\}$ of V a $\binom{1}{0}$-tensor T can be written as $T = T(\widehat{\mathbf{e}}^i)\mathbf{e}_i = T^i\mathbf{e}_i$, with $T^i\mathbf{e}_i = a^i\mathbf{e}_i = \mathbf{a}$. For every $\widehat{\mathbf{y}} \in V^*$ holds as known

$$T(\widehat{\mathbf{y}}) = T(y_i\widehat{\mathbf{e}}^i) = y_i T(\widehat{\mathbf{e}}^i) = T(\widehat{\mathbf{e}}^i) < \widehat{\mathbf{y}}, \mathbf{e}_i > = < \widehat{\mathbf{y}}, a^i\mathbf{e}_i > = < \widehat{\mathbf{y}}, \mathbf{a} > .$$

Definition 2.8.4 The numbers $T^i = T(\widehat{\mathbf{e}}^i)$ are called the contravariant components of the tensor T with respect to the basis $\{\mathbf{e}_i\}$. This explains also the name "contravariant 1-tensor". □

2.8.4 $\binom{0}{1}$-tensor = covariant 1-tensor = covector

Definition 2.8.5 A $\binom{0}{1}$-tensor is a linear transformation of V to $\mathbb{R}$. □

Notice(s): 2.8.2

- Write the tensor as $\mathbf{x} \mapsto F(\mathbf{x})$. In accordeance with Definition 2.2.1 the functions F is a linear function on V and can be written as $\mathbf{x} \mapsto F(\mathbf{x}) \; = < \widehat{\mathbf{f}}, \mathbf{x} >$, for certain $\widehat{\mathbf{f}} \in V^*$. The set of $\binom{0}{1}$-tensors is exactly equal to Dual Space V^* of the Vector Space V, the startingpoint.
- For every basis $\{\mathbf{e}_i\}$ of V a $\binom{0}{1}$-tensor F can be written as $F \;=\; F(\mathbf{e}_i)\widehat{\mathbf{e}}^i \;=\; F_i\widehat{\mathbf{e}}^i$, with $F_i\widehat{\mathbf{e}}^i \;=\; f_i\widehat{\mathbf{e}}^i \;=\; \widehat{\mathbf{f}}$. For every $\mathbf{x} \in V$ holds as known

$$F(\mathbf{x}) \;=\; T(\widehat{\mathbf{x}}^i \mathbf{e}_i) \;=\; \widehat{\mathbf{x}}^i\, F(\mathbf{e}_i) \;=\; F(\mathbf{e}_i) < \widehat{\mathbf{e}}^i, \mathbf{x} > = < f_i\,\widehat{\mathbf{e}}^i, \mathbf{x} > = < \widehat{\mathbf{f}}, \mathbf{x} > .$$

Definition 2.8.6 The numbers $F_i \;=\; F(\mathbf{e}_i)$ are called the covariant components of the tensor F with respect to the basis $\{\mathbf{e}_i\}$. This explains also the name "covariant 1-tensor". □

2.8.5 $\binom{0}{2}$-tensor = covariant 2-tensor = linear transformation: $V \rightarrow V^*$

Definition 2.8.7 A $\binom{0}{2}$-tensor is a linear transformation of $V \times V$ to $\mathbb{R}$, which is linear in both arguments. A $\binom{0}{2}$-tensor is also called a bilinear function on $V \times V$. □

Clarification(s): 2.8.1

- For a $\binom{0}{2}$-tensor φ holds:

$$\begin{aligned} \varphi(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) &= \alpha\varphi(\mathbf{x}, \mathbf{z}) + \beta\varphi(\mathbf{y}, \mathbf{z}), \\ \varphi(\mathbf{x}, \alpha\mathbf{y} + \beta\mathbf{z}) &= \alpha\varphi(\mathbf{x}, \mathbf{y}) + \beta\varphi(\mathbf{x}, \mathbf{z}), \end{aligned}$$

  for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and for every $\alpha, \beta \in \mathbb{R}$.

Definition 2.8.8 For every pair of $\binom{0}{2}$-tensors φ and ψ, the $\binom{0}{2}$-tensor $\varphi + \psi$ is defined by $(\varphi + \psi)(\mathbf{x}, \mathbf{y}) \;=\; \varphi(\mathbf{x}, \mathbf{y}) + \psi(\mathbf{x}, \mathbf{y})$ and for every $\alpha \in \mathbb{R}$ the $\binom{0}{2}$-tensor $\alpha\varphi$ is defined by $(\alpha\varphi)(\mathbf{x}, \mathbf{y}) \;=\; \alpha\varphi(\mathbf{x}, \mathbf{y})$. □

Comment(s): 2.8.2

- The set of $\binom{0}{2}$-tensors is a Vector Space over $\mathbb{R}$, which is notated by $V^* \otimes V^*$ and also with $T^0_2(V)$.

Example(s): 2.8.1 Every type of inner product on V, such as considered in Section 2.5, is a $\binom{0}{2}$-tensor. If there is made a fixed choice for an inner product on some Vector Space V, then this inner product is called a fundamental tensor. □

Definition 2.8.9 For every $\widehat{\mathbf{p}}, \widehat{\mathbf{q}} \in V^*$ the $\binom{0}{2}$-tensor $\widehat{\mathbf{p}} \otimes \widehat{\mathbf{q}}$ on V is defined by

$$(\widehat{\mathbf{p}} \otimes \widehat{\mathbf{q}})(\mathbf{x}, \mathbf{y}) = < \widehat{\mathbf{p}}, \mathbf{x} >< \widehat{\mathbf{q}}, \mathbf{y} > .$$

□

Notice(s): 2.8.3

- If the system $\{\widehat{\mathbf{p}}, \widehat{\mathbf{q}}\}$ is linear independent then $\widehat{\mathbf{p}} \otimes \widehat{\mathbf{q}} \neq \widehat{\mathbf{q}} \otimes \widehat{\mathbf{p}}$.

Definition 2.8.10 A linear transformation $\mathcal{K} : V \to V^*$ is associated with a $\binom{0}{2}$-tensor on two manners:

$$K(\mathbf{x}, \mathbf{y}) = < \mathcal{K}\mathbf{x}, \mathbf{y} >,$$
$$\kappa(\mathbf{x}, \mathbf{y}) = < \mathcal{K}\mathbf{y}, \mathbf{x} >$$

□

If there are made good compromises then there exists an 1-1 correspondence between the $\binom{0}{2}$-tensors and the linear transformations from V to V^*.

Theorem 2.8.1 Given: a $\binom{0}{2}$-tensor K.

- There exists just one linear transformation $\mathcal{K} : V \to V^*$ such that

$$\forall \mathbf{x} \in V\, \forall \mathbf{y} \in V : K(\mathbf{x}, \mathbf{y}) \; = < \mathcal{K}\mathbf{x}, \mathbf{y} > .$$

  Explicitly: $\mathcal{K} \;=\; K(\cdot, \mathbf{e}_i)\widehat{\mathbf{e}}^i$, so $\mathcal{K}\mathbf{u} \;=\; K(\mathbf{u}, \mathbf{e}_i)\widehat{\mathbf{e}}^i$.
- There exists just one linear transformation $\mathcal{K}^* : V \to V^*$ such that

$$\forall \mathbf{x} \in V\, \forall \mathbf{y} \in V : K(\mathbf{x}, \mathbf{y}) \; = < \mathcal{K}^*\mathbf{y}, \mathbf{x} > .$$

  Explicitly: $\mathcal{K}^* \;=\; K(\mathbf{e}_i, \cdot)\widehat{\mathbf{e}}^i$, so $\mathcal{K}^*\mathbf{w} \;=\; K(\mathbf{e}_i, w)\widehat{\mathbf{e}}^i$. □

Proof

- Choose a fixed $\mathbf{a} \in V$ and look to the $\binom{0}{1}$-tensor $\mathbf{x} \mapsto K(\mathbf{a}, \mathbf{x})$. Interpret anyhow $\mathbf{a}$ as variable, such that there is defined a linear transformation $\mathcal{K} : V \to V^*$ by $\mathbf{a} \mapsto K(\mathbf{a}, \cdot) \; = < \mathcal{K}\mathbf{a}, \cdot >$. Then $K(\mathbf{u}, \mathbf{v}) \; = < \mathcal{K}\mathbf{u}, \mathbf{v} >$.
  The $\binom{0}{1}$-tensor $K(\mathbf{u}, \mathbf{x})$ can be written as $K(\mathbf{u}, \mathbf{x}) \;=\; K(\mathbf{u}, \mathbf{e}_i) < \widehat{\mathbf{e}}^i, \mathbf{x} >$, see Notice 2.8.2.
  After a basis transition holds $\mathcal{K}\mathbf{u} \;=\; K(\mathbf{u}, \mathbf{e}_{i'})\widehat{\mathbf{e}}^{i'}$.
- Choose a fixed $\mathbf{a} \in V$ and look to the $\binom{0}{1}$-tensor $\mathbf{x} \mapsto K(\mathbf{x}, \mathbf{a})$. Define the linear transformation $\mathcal{K}^* : V \to V^*$ by $\mathbf{a} \mapsto K(\cdot, \mathbf{a}) \; = < \mathcal{K}^*\mathbf{a}, \cdot >$. Then $K(\mathbf{u}, \mathbf{v}) \; = < \mathcal{K}^*\mathbf{v}, \mathbf{u} >$.
  The $\binom{0}{1}$-tensor $K(\mathbf{x}, \mathbf{w})$ can be written as $K(\mathbf{x}, \mathbf{w}) \;=\; K(\mathbf{e}_i, \mathbf{w}) < \widehat{\mathbf{e}}^i, \mathbf{w} >$, see Notice 2.8.2, so $\mathcal{K}^*\mathbf{w} \;=\; K(\mathbf{e}_i, w)\widehat{\mathbf{e}}^i$.
  The explicit representation after a basis transition holds $\mathcal{K}^*\mathbf{u} \;=\; K(\mathbf{e}_{i'}, \mathbf{u})\widehat{\mathbf{e}}^{i'}$. □

Notice(s): 2.8.4

- If $\forall \mathbf{x} \in V\, \forall \mathbf{y} \in V : \; K(\mathbf{x}, \mathbf{y}) \;=\; K(\mathbf{y}, \mathbf{x})$ then holds $\mathcal{K} \;=\; \mathcal{K}^*$.
- If $\forall \mathbf{x} \in V\, \forall \mathbf{y} \in V : \; K(\mathbf{x}, \mathbf{y}) \;=\; -K(\mathbf{y}, \mathbf{x})$ then holds $\mathcal{K} \;=\; -\mathcal{K}^*$.
- To applications there is often said that a 2-tensor is the *same* as a matrix. There enough arguments to bring up against that suggestion. After the choice of a basis a 2-tensor can be *represented* by a matrix, on the same way as is done to linear transformations.

Definition 2.8.11 Let $\{\mathbf{e}_i\}$ a basis of V and φ a $\binom{0}{2}$-tensor on V. The numbers φ_{ij}, defined by $\varphi_{ij} \;=\; \varphi(\mathbf{e}_i, \mathbf{e}_j)$, are called the covariant components of the tensor φ with respect to the basis $\{\mathbf{e}_i\}$. □

Notice(s): 2.8.5

- The addition "covariant" to (covariant) components is here more a verbiage. There are no other components then covariant components such that only vectors can be filled in. See nevertheless also Paragraph 2.8.11.
- For $\mathbf{x}, \mathbf{y} \in V$ holds $\varphi(\mathbf{x}, \mathbf{y}) = x^i y^j \varphi_{ij}$.
- The action of a $\binom{0}{2}$-tensor on two vectors $\mathbf{x}$ and $\mathbf{y}$ can be written as $\varphi(\mathbf{x}, \mathbf{y}) = \varphi_{ij} x^i y^j$.
- Let $\{\mathbf{e}_{i'}\}$ be a second basis on V then holds $\varphi_{i'j'} = \varphi(\mathbf{e}_{i'}, \mathbf{e}_{j'}) = \varphi(A^i_{i'}\mathbf{e}_i, A^j_{j'}\mathbf{e}_j) = A^i_{i'} A^j_{j'} \varphi_{ij}$. Hereby the relation is denoted between the covariant components of a tensor for two arbitrary bases. Compare with Paragraph 2.4.

Definition 2.8.12 Let $\{\mathbf{e}_i\}$ be basis on V. To every pair indices i and j the $\binom{0}{2}$-tensor $\widehat{\mathbf{e}}^i \otimes \widehat{\mathbf{e}}^j$ is defined by

$$(\widehat{\mathbf{e}}^i \otimes \widehat{\mathbf{e}}^j)(\mathbf{x}, \mathbf{y}) = \widehat{\mathbf{e}}^i(\mathbf{x}) \widehat{\mathbf{e}}^j(\mathbf{y}) = < \widehat{\mathbf{e}}^i, \mathbf{x} >< \widehat{\mathbf{e}}^j, \mathbf{y} > .$$

□

Lemma 2.8.1

- The set $\{\widehat{\mathbf{e}}^i \otimes \widehat{\mathbf{e}}^j\}$ is a basis of $T^0_2(V)$. There holds: $\varphi = \varphi_{ij} \widehat{\mathbf{e}}^i \otimes \widehat{\mathbf{e}}^j$.
- If $\dim(\, V\,) = n$ then $\dim(\, T^0_2(V)\,) = n^2$. □

Proof Let $\varphi \in T^0_2(V)$ then for all $\mathbf{x}, \mathbf{y} \in V$ holds that

$$\varphi(\mathbf{x}, \mathbf{y}) = \varphi(x^i \mathbf{e}_i, y^j \mathbf{e}_j) = \varphi_{ij} \varphi(x^i) \varphi(y^j) = \varphi_{ij} < \widehat{\mathbf{e}}^i, \mathbf{x} >< \widehat{\mathbf{e}}^j, \mathbf{y} > = \varphi_{ij} (\widehat{\mathbf{e}}^i \otimes \widehat{\mathbf{e}}^j)(\mathbf{x}, \mathbf{y}),$$

or $\varphi = \varphi_{ij} \widehat{\mathbf{e}}^i \otimes \widehat{\mathbf{e}}^j$. The Vector Space $T^0_2(V)$ is accordingly spanned by the set $\{\widehat{\mathbf{e}}^i \otimes \widehat{\mathbf{e}}^j\}$. The final part to prove is that the system $\{\widehat{\mathbf{e}}^i \otimes \widehat{\mathbf{e}}^j\}$ is linear independent. Suppose that there are n^2 numbers α_{ij} such that $\alpha_{ij} \widehat{\mathbf{e}}^i \otimes \widehat{\mathbf{e}}^j = 0$. For every k and l holds that

$$0 = \alpha_{ij} (\widehat{\mathbf{e}}^i \otimes \widehat{\mathbf{e}}^j)(\mathbf{e}_k, \mathbf{e}_l) = \alpha_{ij} \delta^i_k \delta^j_l = \alpha_{kl}.$$

□

Comment(s): 2.8.3 As previously stated an inner product is an $\binom{0}{2}$-tensor. Hereby is considered a linear transformation from V to V^*, which is defined by $\mathbf{a} \mapsto (\mathbf{a}, \cdot)$. That is exactly the bijective linear transformation $\mathcal{G}$ out of Theorem 2.5.1.

2.8.6 $\binom{2}{0}$-tensor = contravariant 2-tensor = linear transformation: $V^* \to V$

Definition 2.8.13 A $\binom{2}{0}$-tensor is a linear transformation of $V^* \times V^*$ to $\mathbb{R}$, which is linear in both arguments. A $\binom{2}{0}$-tensor is also called a bilinear function on $V^* \times V^*$. $\square$

Clarification(s): 2.8.2

- For a $\binom{2}{0}$-tensor H holds:

$$\begin{aligned} H(\alpha\widehat{\mathbf{x}} + \beta\widehat{\mathbf{y}}, \widehat{\mathbf{z}}) &= \alpha H(\widehat{\mathbf{x}}, \widehat{\mathbf{z}}) + \beta H(\widehat{\mathbf{y}}, \widehat{\mathbf{z}}), \\ H(\widehat{\mathbf{x}}, \alpha\widehat{\mathbf{y}} + \beta\widehat{\mathbf{z}}) &= \alpha H(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) + \beta H(\widehat{\mathbf{x}}, \widehat{\mathbf{z}}), \end{aligned}$$

  for all $\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}} \in V^*$ and for every $\alpha, \beta \in \mathbb{R}$.

Definition 2.8.14 For every pair of $\binom{2}{0}$-tensors H and h, the $\binom{2}{0}$-tensor $H + h$ is defined by $(H + h)(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = H(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) + h(\widehat{\mathbf{x}}, \widehat{\mathbf{y}})$ and for every $\alpha \in \mathbb{R}$ the $\binom{2}{0}$-tensor αH is defined by $(\alpha H)(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = \alpha H(\widehat{\mathbf{x}}, \widehat{\mathbf{y}})$. $\square$

Comment(s): 2.8.4

- The set of $\binom{2}{0}$-tensors is a Vector Space over $\mathbb{R}$, which is notated by $V \otimes V$ and also with $T^2_0(V)$.

Definition 2.8.15 For every $\mathbf{x}, \mathbf{y} \in V$ the $\binom{2}{0}$-tensor $\mathbf{x} \otimes \mathbf{y}$ on V is defined by

$$(\mathbf{x} \otimes \mathbf{y})(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}) = <\widehat{\mathbf{u}}, \mathbf{x}><\widehat{\mathbf{v}}, \mathbf{y}> .$$

$\square$

Notice(s): 2.8.6

- If the system $\{\mathbf{x}, \mathbf{y}\}$ is linear independent then $\mathbf{x} \otimes \mathbf{y} \neq \mathbf{y} \otimes \mathbf{x}$.

Definition 2.8.16 A linear transformation $\mathcal{H} : V^* \to V$ is associated with a $\binom{2}{0}$-tensor on two manners:

$$H(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = < \widehat{\mathbf{x}}, \mathcal{H}\widehat{\mathbf{y}} >,$$
$$h(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = < \widehat{\mathbf{y}}, \mathcal{H}\widehat{\mathbf{x}} >$$

□

If there are made good compromises then there exists an 1-1 correspondence between the $\binom{2}{0}$-tensors and the linear transformations from V^* to V.

Theorem 2.8.2 Given: a $\binom{2}{0}$-tensor H.

- There exists just one linear transformation $\mathcal{H} : V^* \to V$ such that

$$\forall \widehat{\mathbf{x}} \in V^* \, \forall \widehat{\mathbf{y}} \in V^* : H(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = < \widehat{\mathbf{x}}, \mathcal{H}\widehat{\mathbf{y}} > .$$

  Explicitly: $\mathcal{H} = H(\widehat{\mathbf{e}}^i, \cdot)\mathbf{e}_i$, so $\mathcal{H}\widehat{\mathbf{v}} = H(\widehat{\mathbf{e}}^i, \widehat{\mathbf{v}})\mathbf{e}_i$.
- There exists just one linear transformation $\mathcal{H}^* : V^* \to V$ such that

$$\forall \widehat{\mathbf{x}} \in V^* \, \forall \widehat{\mathbf{y}} \in V^* : H(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = < \widehat{\mathbf{y}}, \mathcal{H}^*\widehat{\mathbf{x}} > .$$

  Explicitly: $\mathcal{H}^* = H(\cdot, \widehat{\mathbf{e}}^i)\mathbf{e}_i$, so $\mathcal{H}^*\widehat{\mathbf{v}} = H(\widehat{\mathbf{v}}, \widehat{\mathbf{e}}^i)\mathbf{e}_i$. □

□

Proof

- Choose a fixed $\widehat{\mathbf{b}} \in V^*$ and look to the $\binom{1}{0}$-tensor $\widehat{\mathbf{x}} \mapsto H(\widehat{\mathbf{x}}, \widehat{\mathbf{b}})$. Interpret anyhow $\widehat{\mathbf{b}}$ as variable, such that there is defined a linear transformation $\mathcal{H} : V^* \to V$ by $\widehat{\mathbf{b}} \mapsto H(\cdot, \widehat{\mathbf{b}}) = < \cdot, \mathcal{H}\widehat{\mathbf{b}} >$. Then $H(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}) = < \widehat{\mathbf{u}}, \mathcal{H}\widehat{\mathbf{v}} >$.
  See Paragraph 2.8.3 for the explicit notation of $\binom{1}{0}$-tensor.
  After a basis transition holds $\mathcal{H}\widehat{\mathbf{u}} = H(\widehat{\mathbf{e}}^{i'}, \widehat{\mathbf{u}})\,\mathbf{e}_{i'}$.
- Choose a fixed $\widehat{\mathbf{a}} \in V^*$ and look to the $\binom{1}{0}$-tensor $\widehat{\mathbf{x}} \mapsto H(\widehat{\mathbf{a}}, \widehat{\mathbf{x}})$. Interpret anyhow $\widehat{\mathbf{a}}$ as variable, such that there is defined a linear transformation $\mathcal{H} : V^* \to V$ by $\widehat{\mathbf{a}} \mapsto H(\widehat{\mathbf{b}}, \cdot) = < \cdot, \mathcal{H}^*\widehat{\mathbf{b}} >$. Then $H(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}) = < \widehat{\mathbf{v}}, \mathcal{H}^*\widehat{\mathbf{u}} >$.
  See Paragraph 2.8.3 for the explicit notation of $\binom{1}{0}$-tensor.
  After a basis transition holds $\mathcal{H}^*\widehat{\mathbf{u}} = H(\widehat{\mathbf{u}}, \widehat{\mathbf{e}}^{i'})\,\mathbf{e}_{i'}$. Compare with Paragraph 2.4. □

Notice(s): 2.8.7

- If $\forall \widehat{\mathbf{x}} \in V^* \, \forall \widehat{\mathbf{y}} \in V^* : H(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = H(\widehat{\mathbf{y}}, \widehat{\mathbf{x}})$ then holds $\mathcal{H} = \mathcal{H}^*$.
- If $\forall \widehat{\mathbf{x}} \in V^* \, \forall \widehat{\mathbf{y}} \in V^* : H(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = -H(\widehat{\mathbf{y}}, \widehat{\mathbf{x}})$ then holds $\mathcal{H} = -\mathcal{H}^*$.

Definition 2.8.17 Let $\{\mathbf{e}_i\}$ be a basis of V and H a $\binom{2}{0}$-tensor on V. The numbers H^{ij}, defined by $H^{ij} = H(\widehat{\mathbf{e}}^{i'}, \widehat{\mathbf{e}}^{j'})$, are called the <u>contravariant components</u> of the tensor H with respect to the basis $\{\mathbf{e}_i\}$. □

Notice(s): 2.8.8

- The addition "contravariant" to (contravariant) components is here more a verbiage. See nevertheless also Paragraph 2.8.11.
- For $\widehat{\mathbf{x}}, \widehat{\mathbf{y}} \in V$ holds $H(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = x_i y_j H^{ij}$.
- The action of a $\binom{2}{0}$-tensor on two covectors $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ can be written as $H(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = H^{ij} x_i y_j$.
- Let $\{\mathbf{e}_{i'}\}$ be a second basis on V then holds $H^{i'j'} = H(\widehat{\mathbf{e}}^{i'}, \widehat{\mathbf{e}}^{j'}) = H(A^{i'}_i \widehat{\mathbf{e}}^i, A^{j'}_j \widehat{\mathbf{e}}^j) = A^{i'}_i A^{j'}_j H^{ij}$. Hereby the relation is denoted between the (contra)variant components of a $\binom{2}{0}$-tensor for two arbitrary bases.

Definition 2.8.18 Let $\{\mathbf{e}_i\}$ be basis on V. To every pair indices i and j the $\binom{2}{0}$-tensor $\mathbf{e}_i \otimes \mathbf{e}_j$ is defined by

$$(\mathbf{e}_i \otimes \mathbf{e}_j)(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = < \widehat{\mathbf{x}}, \mathbf{e}_i >< \widehat{\mathbf{y}}, \mathbf{e}_j > .$$

□

Lemma 2.8.2

- The set $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ is a basis of $T^2_0(V)$. There holds: $H = H^{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$.
- If $\dim(V) = n$ then $\dim(T^2_0(V)) = n^2$. □

Proof Let $\theta \in T^2_0(V)$ then for all $\widehat{\mathbf{x}}, \widehat{\mathbf{y}} \in V^*$ holds that

$$\theta(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = \theta(x_i \widehat{\mathbf{e}}^i, y_j \widehat{\mathbf{e}}^j) = \theta^{ij} x_i y_j = \theta^{ij} < \widehat{\mathbf{x}}, \mathbf{e}_i >< \widehat{\mathbf{y}}, \mathbf{e}_j >= \theta^{ij} (\mathbf{e}_i \otimes \mathbf{e}_j)(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}),$$

or $\theta = \theta^{ij}\,\mathbf{e}_i \otimes \mathbf{e}_j$.
The Vector Space $T^2_0(V)$ is accordingly spanned by the set $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$. The final part to prove is that the system $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ is linear independent. This part is done similarly as in Lemma 2.8.1. □

2.8.7 $\binom{1}{1}$-tensor = mixed 2-tensor = linear transformation: $V \to V$ and $V^* \to V^*$

Definition 2.8.19 A $\binom{1}{1}$-tensor is a linear transformation of $V^* \times V$ to $\mathbb{R}$, which is linear in both arguments. A $\binom{1}{1}$-tensor is also called a bilinear function on $V^* \times V$. □

Clarification(s): 2.8.3

- For a $\binom{1}{1}$-tensor R holds:

$$
\begin{aligned}
R(\alpha\,\widehat{\mathbf{x}} + \beta\,\widehat{\mathbf{y}}, \mathbf{z}) &= \alpha R(\widehat{\mathbf{x}}, \mathbf{z}) + \beta R(\widehat{\mathbf{y}}, \mathbf{z}), \\
R(\widehat{\mathbf{x}}, \alpha\,\mathbf{y} + \beta\,\mathbf{z}) &= \alpha R(\widehat{\mathbf{x}}, \mathbf{y}) + \beta R(\widehat{\mathbf{x}}, \mathbf{z}),
\end{aligned}
$$

  for all $\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \in V^*$, $\mathbf{y}, \mathbf{z}, \in V$ and for every $\alpha, \beta \in \mathbb{R}$.

Comment(s): 2.8.5

- The set of $\binom{1}{1}$-tensors is a Vector Space over $\mathbb{R}$, which is notated by $V \otimes V^*$ and also with $T^1_1(V)$.

Definition 2.8.20 For every pair $\mathbf{x} \in V$, $\widehat{\mathbf{y}} \in V*$, the $\binom{1}{1}$-tensor $\mathbf{x} \otimes \widehat{\mathbf{y}}$ on V is defined by

$$
(\mathbf{x} \otimes \widehat{\mathbf{y}})(\widehat{\mathbf{u}}, \mathbf{v}) \;=< \widehat{\mathbf{u}}, \mathbf{x} >< \widehat{\mathbf{y}}, \mathbf{v} > .
$$

□

Definition 2.8.21

- With a linear transformation $\mathcal{R} : V \mapsto V$ is associated a $\binom{1}{1}$-tensor:

$$R(\widehat{\mathbf{x}}, \mathbf{y}) = < \widehat{\mathbf{x}}, \mathcal{R}\mathbf{y} > .$$

- With a linear transformation $\mathcal{P} : V^* \mapsto V^*$ is associated a $\binom{1}{1}$-tensor:

$$P(\widehat{\mathbf{x}}, \mathbf{y}) = < \mathcal{P}\widehat{\mathbf{x}}, \mathbf{y} > .$$

□

There exists an 1-1 correspondence between the $\binom{1}{1}$-tensors and the linear transformations from V to V. There exists an 1-1 correspondence between the $\binom{1}{1}$-tensors and the linear transformations from V^* to V^*.

Theorem 2.8.3 Given: a $\binom{1}{1}$-tensor R.

- There exists just one linear transformation $\mathcal{R} : V \rightarrow V$ such that

$$\forall \widehat{\mathbf{x}} \in V^* \, \forall \mathbf{y} \in V : R(\widehat{\mathbf{x}}, \mathbf{y}) = < \widehat{\mathbf{x}}, \mathcal{R}\,\mathbf{y} > .$$

Explicitly: $\mathcal{R} = R(\widehat{\mathbf{e}}^{i}, \cdot)\mathbf{e}_i$, so $\mathcal{R}\,\mathbf{v} = R(\widehat{\mathbf{e}}^{i}, \mathbf{v})\mathbf{e}_i$.
- There exists just one linear transformation $\mathcal{R}^* : V^* \rightarrow V^*$ such that

$$\forall \widehat{\mathbf{x}} \in V^* \, \forall \mathbf{y} \in V : R(\widehat{\mathbf{x}}, \mathbf{y}) = < \mathcal{R}^*\widehat{\mathbf{x}}, \mathbf{y} > .$$

Explicitly: $\mathcal{R}^* = R(\cdot, \mathbf{e}_j)\widehat{\mathbf{e}}^{j}$, so $\mathcal{R}^*\widehat{\mathbf{u}} = R(\widehat{\mathbf{u}}, \mathbf{e}_j)\widehat{\mathbf{e}}^{j}$. □

□

Proof

- Choose a fixed $\mathbf{a} \in V$ and look to the $\binom{1}{0}$-tensor $\widehat{\mathbf{x}} \mapsto R(\widehat{\mathbf{x}}, \mathbf{a})$. Interpret anyhow $\mathbf{a}$ as variable, such that there is defined a linear transformation $\mathcal{R} : V \rightarrow V$ by $\mathbf{a} \mapsto R(\cdot, \mathbf{a}) = < \cdot, \mathcal{R}\,\mathbf{a} > = \mathcal{R}\mathbf{a}$.
  After a basis transition holds $\mathcal{R}\mathbf{u} = R(\widehat{\mathbf{e}}^{i'}, \widehat{\mathbf{u}})\,\mathbf{e}_{i'}$, such that the representation is independent for basis transistions.
- Choose a fixed $\widehat{\mathbf{b}} \in V^*$ and look to the $\binom{0}{1}$-tensor $\mathbf{y} \mapsto R(\widehat{\mathbf{b}}, \mathbf{y})$, an element of V^*. Interpret anyhow $\widehat{\mathbf{b}}$ as variable, such that there is defined a linear transformation $\mathcal{R}^* : V^* \rightarrow V^*$ by $\widehat{\mathbf{b}} \mapsto R(\widehat{\mathbf{b}}, \cdot) = < \mathcal{R}^*\widehat{\mathbf{b}}, \cdot >$. Then $R(\widehat{\mathbf{u}}, \mathbf{v}) = < \mathcal{R}^*\widehat{\mathbf{v}}, \mathbf{u} >$.
  After a basis transition holds $\mathcal{R}^*\widehat{\mathbf{u}} = R(\widehat{\mathbf{u}}, \mathbf{e}_j)\widehat{\mathbf{e}}^{j}$. Compare with Paragraph 2.4. □

Notice(s): 2.8.9

- Shifting $\widehat{\mathbf{x}}$ and y in $R(\widehat{\mathbf{x}}, y)$ gives a meaningless expression.

Definition 2.8.22 Let $\{\mathbf{e}_i\}$ be a basis of V and R a $\binom{1}{1}$-tensor on V. The numbers R^{ij}, defined by $R^{ij} = R(\widehat{\mathbf{e}}^i, \mathbf{e}_j)$, are called the (mixed) components of the tensor R with respect to the basis $\{\mathbf{e}_i\}$. □

Notice(s): 2.8.10

- The addition "mixed" to (contravariant) components is here more a verbiage. There is nothing else that can be done. See nevertheless also Paragraph 2.8.11.
- For $\widehat{\mathbf{x}}, \widehat{\mathbf{y}} \in V$ holds $R(\widehat{\mathbf{x}}, \mathbf{y}) = x_i y^j R^i_j$.
- Let $\{\mathbf{e}_{i'}\}$ be a second basis on V then holds $R^{i'j'} = R(\widehat{\mathbf{e}}^{i'}, \mathbf{e}_{j'}) = R(A^{i'}_i \widehat{\mathbf{e}}^i, A^j_{j'} \mathbf{e}_j) = A^{i'}_i A^j_{j'} R^{ij}$. Hereby the relation is denoted between the (mixed) components of a $\binom{1}{1}$-tensor for two arbitrary bases.

Definition 2.8.23 Let $\{\mathbf{e}_i\}$ be basis on V. To every pair indices i and j the $\binom{1}{1}$-tensor $\mathbf{e}_i \otimes \widehat{\mathbf{e}}_j$ is defined by

$$(\mathbf{e}_i \otimes \widehat{\mathbf{e}}_j)(\widehat{\mathbf{x}}, \mathbf{y}) = < \widehat{\mathbf{x}}, \mathbf{e}_i >< \widehat{\mathbf{e}}^j, \mathbf{y} > .$$

□

Lemma 2.8.3

- The set $\{\mathbf{e}_i \otimes \widehat{\mathbf{e}}^j\}$ is a basis of $T^1_1(V)$. There holds: $R = R^i_j \mathbf{e}_i \otimes \widehat{\mathbf{e}}^j$.
- If $\dim(V) = n$ then $\dim(T^1_1(V)) = n^2$. □

Proof Let $\psi \in T^1_1(V)$ then for all $\widehat{\mathbf{x}} \in V^*, \mathbf{y} \in V$ holds that

$$\psi(\widehat{\mathbf{x}}, \mathbf{y}) = \psi(x_i \widehat{\mathbf{e}}^i, y^j \mathbf{e}_j) = \psi^i_j x_i y^j = \psi^i_j < \widehat{\mathbf{x}}, \mathbf{e}_i >< \widehat{\mathbf{e}}^j, \mathbf{y} >= \psi^i_j (\mathbf{e}_i \otimes \widehat{\mathbf{e}}^j)(\widehat{\mathbf{x}}, \mathbf{y}),$$

or $\psi = \psi^i_j \mathbf{e}_i \otimes \widehat{\mathbf{e}}_j$.

The Vector Space $T^1_1(V)$ is accordingly spanned by the set $\{\mathbf{e}_i \otimes \widehat{\mathbf{e}}_j\}$. The final part to prove is that the system $\{\mathbf{e}_i \otimes \widehat{\mathbf{e}}_j\}$ is linear independent. This part is done similarly as in Lemma 2.8.1. □

2.8.8 $\binom{0}{3}$-tensor = covariant 3-tensor = linear transformation: $V \to (V^* \otimes V^*)$ and $(V \otimes V) \to V^*$

Definition 2.8.24 A $\binom{0}{3}$-tensor is a linear transformation of $V \times V \times V$ to $\mathbb{R}$, which is linear in each of its three vector arguments. □

The meaning is an obvious expansion of the Clarification(s) 2.8.1, 2.8.2 and 2.8.3. See also the general definition 2.8.1.

Definition 2.8.25 For every pair of $\binom{0}{3}$-tensors Ψ and σ, the $\binom{0}{3}$-tensor $\Psi + \sigma$ is defined by $(\Psi + \sigma)(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \Psi(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \sigma(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and for every $\alpha \in \mathbb{R}$ the $\binom{0}{3}$-tensor $\alpha\Psi$ is defined by $(\alpha\Psi)(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \alpha\Psi(\mathbf{x}, \mathbf{y}, \mathbf{z})$. □

Comment(s): 2.8.6

- The set of $\binom{0}{3}$-tensors is a Vector Space over $\mathbb{R}$, which is notated by $V^* \otimes V^* \otimes V^*$ and also with $T^0_3(V)$.

Definition 2.8.26 For every $\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \widehat{\mathbf{w}} \in V^*$ the $\binom{0}{3}$-tensor $\widehat{\mathbf{u}} \otimes \widehat{\mathbf{v}} \otimes \widehat{\mathbf{w}}$ on V is defined by

$$(\widehat{\mathbf{u}} \otimes \widehat{\mathbf{v}} \otimes \widehat{\mathbf{w}})(\mathbf{x}, \mathbf{y}, \mathbf{z}) = <\widehat{\mathbf{u}}, \mathbf{x}><\widehat{\mathbf{v}}, \mathbf{y}><\widehat{\mathbf{w}}, \mathbf{z}> .$$

□

Comment(s): 2.8.7 Also here the order of the tensorproduct is essential.

Definition 2.8.27 Let $\{\mathbf{e}_i\}$ be a basis of V and Ψ a $\binom{0}{3}$-tensor on V. The numbers Ψ_{hij}, defined by $\Psi_{hij} = \Psi(\mathbf{e}_h, \mathbf{e}_i, \mathbf{e}_j)$, are called the covariant components of the tensor Ψ with respect to the basis $\{\mathbf{e}_i\}$. The collection of covariant components of a $\binom{0}{3}$-tensor are organized in a 3-dimensional cubic matrix and is notated by $[\Psi_{hij}]$. □

Lemma 2.8.4

- The set $\{\widehat{\mathbf{e}}^j \otimes \widehat{\mathbf{e}}^j \otimes \widehat{\mathbf{e}}^j\}$ is a basis of $T^0_3(V)$. There holds: $\Psi = \Psi_{hij}\widehat{\mathbf{e}}^j \otimes \widehat{\mathbf{e}}^j \otimes \widehat{\mathbf{e}}^j$.
- If $\dim(V) = n$ then $\dim(T^0_3(V)) = n^3$. □

Comment(s): 2.8.8

- A $\binom{0}{3}$-tensor Ψ can be understood, on 3 different manners as a linear transformation from V to $T^0_2(V) = V^* \otimes V^*$. Thereby on 6 different manners as a linear transformation of V to the "Vector Space of linear transformations $V \rightarrow V^*$ ". Simply said, if there is put a vector $\mathbf{a}$ in a slot of the Tensor Ψ, there is got a $\binom{0}{2}$-tensor, for instance $\Psi(\cdot, \mathbf{a}, \cdot)$. In index notation $\Psi_{hij}\, a^i$.
- A $\binom{0}{3}$-tensor Ψ can be understood, on 6 different manners as a linear transformation from the "Vector Space of linear transformation $V^* \rightarrow V$ " to V^*. Let $\mathcal{H} = H^{ij}\,\mathbf{e}_i \otimes \mathbf{e}_j$. For instance:

$$\Psi(\mathbf{e}_i, \mathbf{e}_j, \cdot) = \Psi(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)\, H^{ij}\widehat{\mathbf{e}}^k = \Psi(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)\, H^{ij} < \widehat{\mathbf{e}}^k, \cdot >$$

  defines a covector. In index notation $\Psi_{ijk}\, H^{ij}$.
- A special case is the stress tensor, which descibes the "stress condition" in an infinity wide linear medium. The stress tensor satisfies the property that $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V : \Psi(\mathbf{x}, \mathbf{y}, \mathbf{z}) = -\Psi(\mathbf{x}, \mathbf{z}, \mathbf{y})$. Consider a parallelogram, spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$. The covector $\widehat{\mathbf{f}} = \Psi(\cdot, \mathbf{a}, \mathbf{b})$ represents the force working at the parallelogram. Indeed, if the vectors $\mathbf{a}$ and $\mathbf{b}$ are changed of order, then the force changes of direction (action=reaction). The linear "stress condition" can be described less general and less elegant, with a 2-tensor. There has to be made use of an inner product and a cross product. That is the way, it is done most of the time in textbooks. See for more information Appendix ??.

2.8.9 $\binom{2}{2}$-tensor = mixed 4-tensor = linear transformation: $(V \rightarrow V) \rightarrow (V \rightarrow V) = \cdots$

Definition 2.8.28 A $\binom{2}{2}$-tensor is a linear transformation of $V^* \times V^* \times V \times V$ to $\mathbb{R}$, which is linear in each of its four vector arguments. □

For more explanation, see Definition 2.8.1.

Comment(s): 2.8.9

- The set of $\binom{2}{2}$-tensors is a Vector Space over $\mathbb{R}$, which is notated by $V \otimes V \otimes V^* \otimes V^*$ and also with $T^2_2(V)$.

Definition 2.8.29 For every set $\mathbf{a}, \mathbf{b} \in V$, $\widehat{\mathbf{c}}, \widehat{\mathbf{d}} \in V^*$ the $\binom{2}{2}$-tensor $\mathbf{a} \otimes \mathbf{b} \otimes \widehat{\mathbf{c}} \otimes \widehat{\mathbf{d}}$ is defined by

$$(\mathbf{a} \otimes \mathbf{b} \otimes \widehat{\mathbf{c}} \otimes \widehat{\mathbf{d}})(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \mathbf{x}, \mathbf{y}) = < \widehat{\mathbf{u}}, \mathbf{a} >< \widehat{\mathbf{v}}, \mathbf{b} >< \widehat{\mathbf{c}}, \mathbf{x} >< \widehat{\mathbf{d}}, \mathbf{y} > .$$

□

Notice(s): 2.8.11

- Also here, if the system $\{\widehat{\mathbf{c}}, \widehat{\mathbf{d}}\}$ is linear independent then $\mathbf{a} \otimes \mathbf{b} \otimes \widehat{\mathbf{c}} \otimes \widehat{\mathbf{d}} \neq \mathbf{a} \otimes \mathbf{b} \otimes \widehat{\mathbf{d}} \otimes \widehat{\mathbf{c}}$.

Definition 2.8.30 Let $\{\mathbf{e}_i\}$ be a basis of V and Θ a $\binom{2}{2}$-tensor on V. The numbers Θ^{jk}_{hi}, defined by $\Theta^{jk}_{hi} = \Theta(\widehat{\mathbf{e}}^j, \widehat{\mathbf{e}}^k, \mathbf{e}_h, \mathbf{e}_i)$, are called the <u>(mixed) components</u> of the tensor Θ with respect to the basis $\{\mathbf{e}_i\}$. The collection of mixed components of a $\binom{2}{2}$-tensor are organized in a <u>4-dimensional cubic matrix</u> and is notated by $[\Theta^{jk}_{hi}]$. □

Lemma 2.8.5

- The set $\{\mathbf{e}_j \otimes \mathbf{e}_k \otimes \widehat{\mathbf{e}}^h \otimes \widehat{\mathbf{e}}^i\}$ is a basis of $T^2_2(V)$. There holds: $\Theta = \Theta^{jk}_{hi}\, \mathbf{e}_j \otimes \mathbf{e}_k \otimes \widehat{\mathbf{e}}^h \otimes \widehat{\mathbf{e}}^i$.
- If $\dim(V) = n$ then $\dim(T^2_2(V)) = n^4$. □

Comment(s): 2.8.10 A $\binom{2}{2}$-tensor can be understood on many different ways as a linear transformation of a "space of linear transformations" to a "space of linear transformations".

- The case: $(V \to V) \to (V \to V)$ and $(V \to V) \to (V^* \to V^*)$.
  Let $\mathcal{R} : V \to V$ be a linear transformation. Write $\mathcal{R} = R^l_m \mathbf{e}_l \otimes \widehat{\mathbf{e}}^m$.
  Form the $\binom{1}{1}$-tensor $R^l_m \Theta(\cdot, \widehat{\mathbf{e}}^m, \mathbf{e}_l, \cdot)$. This tensor can be seen as a linear transformation $V \to V$, or $V^* \to V^*$. The actions of these "image-transformations" on V, respectively V^* are described by:

$$
\begin{aligned}
(\Theta R) : V \to V \quad &: \quad \mathbf{x} \mapsto (\Theta R)\mathbf{x} &&= R^l_m \Theta(\widehat{\mathbf{e}}^j, \widehat{\mathbf{e}}^m, \mathbf{e}_l, \mathbf{x})\mathbf{e}_j \\
& &&= R^{l'}_{m'} \Theta(\widehat{\mathbf{e}}^{j'}, \widehat{\mathbf{e}}^{m'}, \mathbf{e}_{l'}, \mathbf{x})\mathbf{e}_{j'} \\
(\Theta R)^* : V^* \to V^* &: \quad \widehat{\mathbf{x}} \mapsto (\Theta R)\widehat{\mathbf{x}} &&= R^l_m \Theta(\widehat{\mathbf{x}}, \widehat{\mathbf{e}}^m, \mathbf{e}_l, \mathbf{e}_j)\widehat{\mathbf{e}}^j \\
& &&= R^{l'}_{m'} \Theta(\widehat{\mathbf{x}}, \widehat{\mathbf{e}}^{m'}, \mathbf{e}_{l'}, \mathbf{e}_{j'})\widehat{\mathbf{e}}^{j'}
\end{aligned}
$$

  In index notation is written

$$[R^k_h] \mapsto [(\Theta R)^k_h] = [\Theta^{jk}_{hi} R^i_j], \quad [x^k] \mapsto [\Theta^{jk}_{hi} R^i_j x^h], \quad [x_h] \mapsto [\Theta^{jk}_{hi} R^i_j x_k].$$

  This "game" can also be played with summations about other indices.
- The case: $(V \to V^*) \to (V \to V^*)$ and $(V^* \to V) \to (V^* \to V)$.
  Let $\mathcal{K} : V \to V^*$ be a linear transformation. Write $\mathcal{K} = K_{ij} \widehat{\mathbf{e}}^i \otimes \widehat{\mathbf{e}}^j$. In this case there is worked only with the index notation.

$$[K_{hi}] \mapsto [\Theta^{jk}_{hi} K_{jk}], \quad [x^h] \mapsto [\Theta^{jk}_{hi} K_{jk} x^h], \quad \text{other choice: } [x^h] \mapsto [\Theta^{jk}_{hi} K_{jk} x^i]$$

  With $\mathcal{H} : V^* \to V$ and $\mathcal{H} = H^{jk} \mathbf{e}_j \otimes \mathbf{e}_k$,

$$[H^{jk}] \mapsto [\Theta^{jk}_{hi} H^{hi}], \quad [x_j] \mapsto [\Theta^{jk}_{hi} H^{hi} x_j], \quad \text{other choice: } [x_k] \mapsto [\Theta^{jk}_{hi} H^{hi} x_k].$$

Et cetera.

The Hooke-tensor in the linear elasticity theory is an important example of a 4-tensor. This tensor transforms linearly a "deformation condition", described by a linear transformation, to a "stress condition", also described by a linear transformation. See for more information Appendix ??.

2.8.10 Continuation of the general considerations about $\binom{r}{s}$-tensors. Contraction and $\otimes$.

The starting-point is the general definition in paragraph 2.8.1.

Definition 2.8.31 Given: An ordered collection of r vectors $\mathbf{a}, \mathbf{b}, \cdots, \mathbf{d} \in V$. An ordered collection of s covectors $\widehat{\mathbf{p}}, \widehat{\mathbf{q}}, \cdots, \widehat{\mathbf{u}} \in V^*$.
The $\binom{r}{s}$-tensor $\underbrace{\mathbf{a} \otimes \mathbf{b} \otimes \cdots \otimes \mathbf{d}}_{r \text{ vectors}} \otimes \underbrace{\widehat{\mathbf{p}} \otimes \widehat{\mathbf{q}} \otimes \cdots \otimes \widehat{\mathbf{u}}}_{s \text{ covectors}}$ is defined by

$$\Big(\mathbf{a} \otimes \mathbf{b} \otimes \cdots \otimes \mathbf{d} \otimes \widehat{\mathbf{p}} \otimes \widehat{\mathbf{q}} \otimes \cdots \otimes \widehat{\mathbf{u}}\Big)(\underbrace{\widehat{\mathbf{v}}, \widehat{\mathbf{w}}, \cdots, \widehat{\mathbf{z}}}_{r \text{ covectors}}, \underbrace{\mathbf{f}, \mathbf{g}, \cdots, \mathbf{k}}_{s \text{ vectors}}) =$$

$$< \widehat{\mathbf{v}}, \mathbf{a} > \cdots < \widehat{\mathbf{z}}, \mathbf{d} >< \widehat{\mathbf{p}}, \mathbf{f} > \cdots < \widehat{\mathbf{u}}, \mathbf{k} > .$$

For every choice of the covectors and vectors the righthand side is a product of $(r + s)$ real numbers! □

Definition 2.8.32 For every pair of $\binom{r}{s}$-tensors T and t, the $\binom{r}{s}$-tensor $T+t$ is defined by $(T+t)(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}, \cdots, \widehat{\mathbf{z}}, \mathbf{f}, \mathbf{g}, \cdots, \mathbf{k}) = T(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}, \cdots, \widehat{\mathbf{z}}, \mathbf{f}, \mathbf{g}, \cdots, \mathbf{k}) + t(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}, \cdots, \widehat{\mathbf{z}}, \mathbf{f}, \mathbf{g}, \cdots, \mathbf{k})$ and for every $\alpha \in \mathbb{R}$ the $\binom{r}{s}$-tensor αT is defined by $(\alpha T)(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}, \cdots, \widehat{\mathbf{z}}, \mathbf{f}, \mathbf{g}, \cdots, \mathbf{k}) = \alpha T(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}, \cdots, \widehat{\mathbf{z}}, \mathbf{f}, \mathbf{g}, \cdots, \mathbf{k})$. □

The proof of the following theorem goes the same as the foregoing lower order examples.

Theorem 2.8.4

- The set of $\binom{r}{s}$-tensors is a Vector Space over $\mathbb{R}$, which is notated by $T^r_s(V)$.
- Let $\{\mathbf{e}_i\}$ be basis on V and $\dim(V) = n$ then a basis of $T^r_s(V)$ is given by

  $$\{\,\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_r} \otimes \widehat{\mathbf{e}}^{j_1} \otimes \widehat{\mathbf{e}}^{j_2} \otimes \cdots \otimes \widehat{\mathbf{e}}^{j_s}\,\}, \text{ with } \begin{cases} 1 \leq i_1 \leq n, \cdots, 1 \leq i_r \leq n, \\ 1 \leq j_1 \leq n, \cdots, 1 \leq j_s \leq n. \end{cases}$$

  So $\dim T^r_s = n^{r+s}$.
- In the expansion

  $$T = T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_r} \otimes \widehat{\mathbf{e}}^{j_1} \otimes \widehat{\mathbf{e}}^{j_2} \otimes \cdots \otimes \widehat{\mathbf{e}}^{j_s},$$

  the components are given by

  $$T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = T(\widehat{\mathbf{e}}^{i_1}, \widehat{\mathbf{e}}^{i_2}, \cdots, \widehat{\mathbf{e}}^{i_r}, \mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \cdots, \mathbf{e}_{j_s}\,)$$

- If there is changed of a basis the following transition rule holds

  $$T^{i'_1 i'_2 \cdots i'_r}_{j'_1 j'_2 \cdots j'_s} = A^{i'_1}_{i_1} A^{i'_2}_{i_2} \cdots A^{i'_r}_{i_r} A^{j_1}_{j'_1} A^{j_2}_{j'_2} \cdots A^{j_s}_{j'_s} T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$$

□

The order of a tensor can be decreased with 2 points. The definition makes use of a basis of V. The definition is independent of *which* basis is chosen.

Definition 2.8.33 Let $\{\mathbf{e}_i\}$ be basis on V. Let $T \in T^r_s(V)$ with $r \geq 1$ and $s \geq 1$. Consider the summation

$$T(\cdots, \widehat{\mathbf{e}}^{i}, \cdots, \mathbf{e}_i, \cdots) = T(\cdots, \widehat{\mathbf{e}}^{i'}, \cdots, \mathbf{e}_{i'}, \cdots).$$

The dual basis vectors stay on a fixed chosen "covector place". The basis vectors stay on a fixed chosen "vector place". The defined summation is a $\binom{r-1}{s-1}$-tensor. The corresponding linear transformation from $T^r_s(V)$ to $T^{r-1}_{s-2}(V)$ is called a contraction. □

Example(s): 2.8.2

- The contraction of a $\binom{1}{1}$-tensor R is scalar. In index notation: $R^i_i = R^{i'}_{i'}$. This is the "trace of the matrix". In the special case of a $\binom{1}{1}$-tensor of the form $\mathbf{a} \otimes \widehat{\mathbf{b}}$: $b^i a_i = b^{i'} a_{i'} = < \widehat{\mathbf{b}}, \mathbf{a} >$.
- Consider the mixed 5-tensor ϕ, in index notation

$$\phi^{i\cdot}_{\cdot jklm} = g^{hi} \phi_{hjklm}.$$

A contraction over the first two indices (i and j) gives a 3-tensor ψ, the covariant components are given by

$$\psi_{klm} = \phi^{i\cdot}_{\cdot iklm} = g^{hi} \phi_{hiklm} = \phi^{\cdot i}_{i\cdot klm}$$

□

In some special cases a "tensorproduct" is already seen, notated by $\otimes$. In general, if $S \in T^{r_1}_{s_1}(V)$ and $T \in T^{r_2}_{s_2}(V)$ are given, then the producttensor $S \otimes T \in T^{r_1+r_2}_{s_1+s_2}(V)$ can be build. To keep the book-keeping under control, a definition is given for some representative special case.

Definition 2.8.34 Given: $R \in T^2_1(V)$ and $S \in T^1_3(V)$.
Then: $R \otimes S \in T^3_4(V)$ is defined by

$$\left(R \otimes S\right)(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \widehat{\mathbf{w}}, \mathbf{r}, \mathbf{x}, \mathbf{y}, \mathbf{z}) = R(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \mathbf{r})\, S(\widehat{\mathbf{w}}, \mathbf{x}, \mathbf{y}, \mathbf{z})$$

Pay attention to the order of how "the vectors and covectors are filled in"! □

Comment(s): 2.8.11

- For the sake of clarity:

$$\left(S \otimes R\right)(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \widehat{\mathbf{w}}, \mathbf{r}, \mathbf{x}, \mathbf{y}, \mathbf{z}) = S(\widehat{\mathbf{u}}, \mathbf{r}, \mathbf{x}, \mathbf{y})\, R(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}, \mathbf{z}).$$

- The components of the mentioned tensorproducts are to a given basis

$$\left(R \otimes S\right)^{ijk}_{lmnr} = R^{ij}_l S^k_{mnr}, \text{ respectively } \left(S \otimes R\right)^{ijk}_{lmnr} = S^i_{lmn} R^{jk}_r.$$

- The tensorproduct is not commutative, in general: $R \otimes S \neq S \otimes R$.
- The tensorproduct is associative, that means that: $(R \otimes S) \otimes T = R \otimes (S \otimes T)$. Practical it means that the expression $R \otimes S \otimes T$ is useful.

2.8.11 Tensors on Vector Spaces provided with an inner product

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$.
- An inner product $(\cdot, \cdot)$ on V.

There is known out of the paragraphs 2.5 and 2.6 that if there is "chosen an inner product on V", "V and V^* can be identified with each other". There exists a bijective linear transformation $\mathcal{G} : V \rightarrow V^*$ with inverse $\mathcal{G}^{-1} : V^* \rightarrow V$. To every basis $\{\mathbf{e}_i\}$ on V, there is available the associated "reciprocal" basis $\{\mathbf{e}^i\}$ in V, such that $(\mathbf{e}^i, \mathbf{e}_j) = \delta^i_j$.

This means that it is sufficient to work with $\binom{0}{p}$-tensors, the covariant tensors. Every "slot" of some mixed $\binom{r}{s}$-tensor, which is sensitive for covectors can be made sensitive for a vector by transforming such a vector by the linear transformation $\mathcal{G}$. Otherwise every "slot" which is sensitive for vectors can be made sensitive for covectors by using the linear transformation $\mathcal{G}^{-1}$.

Summarized: If there is chosen some fixed inner product, it is enough to speak about p-tensors. Out of every p-tensor there can be constructed some type $\binom{r}{s}$-tensor, with $r + s = p$.

Conclusion(s): (IMPORTANT)
A FIXED CHOSEN inner product $(\cdot, \cdot)$ on V leads to:

- calculations with the usual rules of an inner product, replace all angular hooks $< \cdot, \cdot >$ by round hooks $(\cdot, \cdot)$,
- correct expressions, if the hats $\widehat{\cdot}$ are dropped in all the formules of paragraph 2.8.1 till 2.8.10.

Some explanations by examples.

Example(s): 2.8.3

- A 1-tensor is of the form $\mathbf{x} \mapsto (\mathbf{x}, \mathbf{a})$. The covariant components of $\mathbf{a}$ are $a_j = (\mathbf{a}, \mathbf{e}_j)$. The contravariant components are $a^i = (\mathbf{e}^i, \mathbf{a})$.
- A 2-tensor R has the representation $R(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathcal{R}\mathbf{y}) = (\mathcal{R}^*\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$. Here is $\mathcal{R} : V \to V$ a linear transformation and $\mathcal{R}^*$ is the adjoint transformation of $\mathcal{R}$. There are covariant components $R_{ij} = R(\mathbf{e}_i, \mathbf{e}_j)$, contravariant components $R^{ij} = R(\mathbf{e}^i, \mathbf{e}^j)$ and two types mixed components $R_{i\cdot}^{\cdot k} = R(\mathbf{e}_i, \mathbf{e}^k)$ and $R_{\cdot j}^{l\cdot} = R(\mathbf{e}^l, \mathbf{e}_j)$. There holds $R_{i\cdot}^{\cdot k} = g^{kj}R_{ij} = g_{il}R^{lk}$, et cetera. The covariant components of the image of $\mathcal{R}\mathbf{x}$ are given by $R_{ij}x^j$, et cetera.
- The covariant components of the 4-tensor out of paragraph 2.8.9 are $\Theta_{hijk} = \Theta(\mathcal{G}\mathbf{e}_h, \mathcal{G}\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) = g_{hl}g_{im}\Theta^{lm\cdot\cdot}_{\cdot\cdot jk}$. With the components of $\mathcal{G}$ the indices can be "lifted and lowered", let them go from covariant to contravariant and vice versa. □

Section 2.9 Mathematical interpretation of the "Engineering tensor concept"

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$.

Comment(s): 2.9.1

- Out of the linear Algebra, the 1-dimensional blocks of numbers (the rows and columns) and the 2-dimensional blocks of numbers (the matrices) are well-known. Within the use of these blocks of numbers is already made a difference between upper and lower indices. Here will be considered q-dimensional blocks of numbers with upper, lower or mixed indices. These kind of "super matrices" are also called "holors"[1]. For instance the covariant components of a 4-tensor leads to a 4-dimensional block of numbers with lower indices.

[1] A holor is a mathematical entity that is made up of one or more independent quantities.

Notation(s):

- $T^0_0(\mathbb{R}^n)$ a long-winded notation for $\mathbb{R}$.
- $T^1_0(\mathbb{R}^n) = \mathbb{R}^n$ is the Vector Space of all columns of length n.
- $T^0_1(\mathbb{R}^n) = \mathbb{R}_n$ is the Vector Space of all rows of length n.
- $T^2_0(\mathbb{R}^n) = \mathbb{R}^{n\times n}$ is the Vector Space of all the $n \times n$-matrices with upper indices.
- $T^1_1(\mathbb{R}^n) = \mathbb{R}^n_n$ is the Vector Space of all the $n \times n$-matrices with mixed indices.
- $T^0_2(\mathbb{R}^n) = \mathbb{R}_{n\times n}$ is the Vector Space of all the $n \times n$-matrices with lower indices.
- $T^1_2(\mathbb{R}^n)$ is the Vector Space of all the 3-dimensional cubic matrices with one upper index and two lower indices.
- $T^r_s(\mathbb{R}^n)$, with $r, s \in \{0, 1, 2, \cdots\}$ fixed, is the Vector Space of all $(r + s)$-dimensional holors with s lower indices and r upper indices.

Comment(s): 2.9.2

- The Vector Space $T^r_s(\mathbb{R}^n)$ over $\mathbb{R}$ is of dimension $n^{(r+s)}$ and is isomorf with $\mathbb{R}^{n^{(r+s)}}$. If for instance the indices are lexicographic ordered then an identification with $\mathbb{R}^{n^{(r+s)}}$ can be achieved.

Notation(s):

- The set of all bases of V is notated by $\mathrm{Bas}(V)$.

Comment(s): 2.9.3

- In the following definitions are given alternative definitions of tensors. A tensor will be defined as a transformation of $\mathrm{Bas}(V)$ to $T^r_s(\mathbb{R}^n)$ for certain r and s. This transformation will be such that if the action on one basis is known, that the action on another basis can be calculated with the use of transition matrices. In other words, if the holor with respect to a certain basis is known, then the holors with respect to other bases are known.

Definition 2.9.1 A 0-tensor, $\binom{0}{0}$-tensor is a transformation of $\mathrm{Bas}(V)$ to $T^0_0(\mathbb{R}^n)$ which adds to every basis an unique number. □

Notation(s):

- The notation $T^0_0(V)$ is also used for $\mathbb{R}$.

Definition 2.9.2 A covariant 1-tensor, $\binom{0}{1}$-tensor or covector is a transformation $\mathcal{F} : \mathrm{Bas}(V) \rightarrow T^0_1(\mathbb{R}^n)$ with the property

$$\left.\begin{array}{rcl} \mathcal{F}(\{\mathbf{e}_i\}) & = & [x_j] \\ \mathcal{F}(\{\mathbf{e}_{i'}\}) & = & [x_{j'}] \end{array}\right\} \Rightarrow x_{j'} = A^j_{j'} x_j.$$

□

Comment(s): 2.9.4

- With every covariant 1-tensor corresponds a linear function $\widehat{\mathbf{x}} = x_i \widehat{\mathbf{e}}^i = x_{i'} \widehat{\mathbf{e}}^{i'}$ which is independent of the chosen basis.

Notation(s):

- The notation $T^0_1(V)$ is also used for V^*.

Definition 2.9.3 A contravariant 1-tensor, $\binom{1}{0}$-tensor or vector is a transformation $\mathcal{F} : \mathrm{Bas}(V) \rightarrow T^1_0(\mathbb{R}^n)$ with the property

$$\left.\begin{array}{rcl} \mathcal{F}(\{\mathbf{e}_i\}) & = & [x^j] \\ \mathcal{F}(\{\mathbf{e}_{i'}\}) & = & [x^{j'}] \end{array}\right\} \Rightarrow x^{j'} = A^{j'}_j x^j.$$

□

Comment(s): 2.9.5

- With every contravariant 1-tensor corresponds a vector $\mathbf{x} = x^i \mathbf{e}_i = x^{i'} \mathbf{e}_{i'}$ which is independent of the chosen basis.

Notation(s):

- The notation $T^1_0(V)$ is also used for V.

Definition 2.9.4 A covariant 2-tensor or $\binom{0}{2}$-tensor is a transformation $\mathcal{S} : \mathrm{Bas}(V) \rightarrow T^0_2(\mathbb{R}^n)$ with the property

$$\left.\begin{array}{rcl} \mathcal{S}(\{\mathbf{e}_i\}) & = & [T_{kl}] \\ \mathcal{S}(\{\mathbf{e}_{i'}\}) & = & [T_{k'l'}] \end{array}\right\} \Rightarrow T_{k'l'} = A^k_{k'} A^l_{l'} T_{kl}.$$

□

Comment(s): 2.9.6

- With every covariant 2-tensor corresponds a bilinear function $S = T_{kl}\widehat{\mathbf{e}}^k \otimes \widehat{\mathbf{e}}^l = T_{k'l'}\widehat{\mathbf{e}}^{k'} \otimes \widehat{\mathbf{e}}^{l'}$ which is independent of the chosen basis.

Definition 2.9.5 A contravariant 2-tensor or $\binom{2}{0}$-tensor is a transformation $\mathcal{S} : \mathrm{Bas}(V) \rightarrow T^2_0(\mathbb{R}^n)$ with the property

$$\left.\begin{array}{rcl} \mathcal{S}(\{\mathbf{e}_i\}) & = & [T^{kl}] \\ \mathcal{S}(\{\mathbf{e}_{i'}\}) & = & [T^{k'l'}] \end{array}\right\} \Rightarrow T^{k'l'} = A^{k'}_k A^l_{l'} T^{kl}.$$

□

Comment(s): 2.9.7

- After the choice of an inner product, there exists a correspondence between a contravariant 2-tensor and a bilinear function. The bilinear function belonging to $\mathcal{S}$ is given by $S = T^{kl}\widehat{\mathbf{e}}_k \otimes \widehat{\mathbf{e}}_l = T^{k'l'}\widehat{\mathbf{e}}_{k'} \otimes \widehat{\mathbf{e}}_{l'}$ and is independent of the chosen basis.

Definition 2.9.6 A mixed 2-tensor or $\binom{1}{1}$-tensor is a transformation $\mathcal{S} : \mathrm{Bas}(V) \to T^1_1(\mathbb{R}^n)$ with the property

$$\left.\begin{array}{rcl} \mathcal{S}(\{\mathbf{e}_i\}) & = & [T^k_l] \\ \mathcal{S}(\{\mathbf{e}_{i'}\}) & = & [T^{k'}_{l'}] \end{array}\right\} \Rightarrow T^{k'}_{l'} = A^{k'}_k A^{l'}_l T^k_l.$$

□

Comment(s): 2.9.8

- After the choice of an inner product, there exists a correspondence between a mixed 2-tensor and a bilinear function. The bilinear function belonging to $\mathcal{S}$ is given by $S = T^k_l \widehat{\mathbf{e}}_k \otimes \widehat{\mathbf{e}}^l = T^{k'}_{l'} \widehat{\mathbf{e}}_{k'} \otimes \widehat{\mathbf{e}}^{l'}$ and is independent of the chosen basis.
- Every mixed 2-tensor corresponds with a linear transformation $\mathcal{T} : V \to V$, defined by $\mathcal{T}\mathbf{x} = T^k_l < \widehat{\mathbf{e}}^l, \mathbf{x} > \mathbf{e}_k$ which is independent of the chosen basis. For this correspondence is no inner product needed.

Definition 2.9.7 A covariant p-tensor or $\binom{0}{p}$-tensor is a transformation $\mathcal{S} : \mathrm{Bas}(V) \to T^0_p(\mathbb{R}^n)$ with the property

$$\left.\begin{array}{rcl} \mathcal{S}(\{\mathbf{e}_i\}) & = & [T_{i_1 \cdots i_p}] \\ \mathcal{S}(\{\mathbf{e}_{i'}\}) & = & [T_{i'_1 \cdots i'_p}] \end{array}\right\} \Rightarrow T_{i'_1 \cdots i'_p} = A^{i_1}_{i'_1} \cdots A^{i_p}_{i'_p} T_{i_1 \cdots i_p}.$$

□

Definition 2.9.8 A contravariant q-tensor or $\binom{q}{0}$-tensor is a transformation $\mathcal{S} : \mathrm{Bas}(V) \to T^q_0(\mathbb{R}^n)$ with the property

$$\left.\begin{array}{rcl} \mathcal{S}(\{\mathbf{e}_i\}) & = & [T^{i_1 \cdots i_p}] \\ \mathcal{S}(\{\mathbf{e}_{i'}\}) & = & [T^{i'_1 \cdots i'_p}] \end{array}\right\} \Rightarrow T^{i'_1 \cdots i'_p} = A^{i'_1}_{i_1} \cdots A^{p}_{i_p} T^{i_1 \cdots i_p}.$$

□

Definition 2.9.9 A mixed (r + s)-tensor, contravriant of the order r and covariant of the order s, or $\binom{r}{s}$-tensor is a transformation $\mathcal{S} : \mathrm{Bas}(V) \to T^r_s(\mathbb{R}^n)$ with the property

$$\left.\begin{array}{rcl} \mathcal{S}(\{\mathbf{e}_i\}) & = & [T^{k_1\cdots k_r}_{l_1\cdots l_s}] \\ \mathcal{S}(\{\mathbf{e}_{i'}\}) & = & [T^{k'_1\cdots k'_r}_{l'_1\cdots l'_s}] \end{array}\right\} \Rightarrow T^{k'_1\cdots k'_r}_{l'_1\cdots l'_s} = A^{k'_1}_{k_1}\cdots A^{k'_r}_{k_r} A^{l_1}_{l'_1}\cdots A^{l_s}_{l'_s} T^{k_1\cdots k_r}_{l_1\cdots l_s}.$$

□

Comment(s): 2.9.9

- A $\binom{r}{s}$-tensor is called contravariant of the order r and covariant of the order s.
- The preceding treated mathematical operations, such as addition, scalar multiplication, multiplication and contraction of tensors are calculations which lead to the same new tensor independent whatever basis is used. In other words, the calculations can be done on every arbitrary basis. Such a calculation is called "tensorial". The calculations are invariant under transformations of coordinates.
- To describe a tensor it is enough to give a holor which respect to a certain basis. With the definitions in this paragraph the holors with respect to other bases can be calculated.

Some examples.

Example(s): 2.9.1

- ◦ Condider the transformation F_1 which adds to every basis of V the matrix $[\delta^k_m]$. The question becomes if F_1 is a mixed 2-tensor?
  There are chosen two bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_{i'}\}$ of V and there is assumed that $F_1(\{\mathbf{e}_i\}) = [\delta^k_m]$ and $F_1(\{\mathbf{e}_{i'}\}) = [\delta^{k'}_{m'}]$. Then holds

  $$\delta^{k'}_{m'} = A^{k'}_k A^k_{m'} = A^{k'}_k A^m_{m'} \delta^k_{m'}$$

  and there follows that F_1 is a mixed 2-tensor. This can also be seen in matrix language. The argument is then:
  There holds $I^{'}_{,} = A^{'} I A_{,}$ for every invertible $n \times n$-matrix $A_{,}$.
  F_1 is the Kronecker tensor.
  ◦ Condider the transformation F_2 which adds to every basis of V the matrix $[\delta_{km}]$. The question becomes if F_2 is a covariant 2-tensor?
  Holds "$I_{,,} = (A_{,})^T I A_{,}$" forevery invertible $n \times n$-matrix? The answer is "no", so F_2 is not a covariant 2-tensor.
  ◦ Condider the transformation F_3 which adds to every basis of V the matrix $[\delta^{km}]$. The question becomes if F_3 is a contravariant 2-tensor?
  answer is "no", because "$I^{''} = A^{'} I (A^{'})^T$" is not valid for every invertible $n \times n$-matrix.
  ◦ If there should be a restriction to orthogonal transition matrices then F_2 and F_3 should be 2-tensors.
  The to the mixed 2-tensor F_1 belonging linear transformation from V to V is given by $\mathbf{x} \mapsto <\widehat{\mathbf{e}}^i, \mathbf{x}> \mathbf{e}_i = x^i \mathbf{e}_i = \mathbf{x}$, the identitiy map on V.
- Consider the transformation F which adds to every basis of V the matrix $\mathrm{diag}(2,1,1)$. The question becomes if F is a covariant, contravariant or a mixed 2-tensor? It is not difficult to find an invertible matrix A such that $\mathrm{diag}(2,1,1) \neq A^{-1} \mathrm{diag}(2,1,1) A$. So it follows immediately that F is not a 2-tensor of the type as asked. □

Example(s): 2.9.2

- ∘ Consider a mixed 2-tensor ϕ. Write

$$\phi(\{\mathbf{e}_i\}) = [q^k_l] = Q.$$

Is the transformation $\alpha : \mathrm{Bas}(V) \rightarrow \mathbb{R}$, defined by

$$\alpha(\{\mathbf{e}_i\}) = q^l_l = \mathrm{trace}(Q),$$

a scalar? If $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_{i'}\}$ are two arbitrary basis of V and $\phi(\{\mathbf{e}_i\}) = q^k_l$ and $\phi(\{\mathbf{e}_{i'}\}) = q^{k'}_{l'}$ then holds

$$q^{k'}_{l'} = A^l_{l'} A^{k'}_k q^k_l,$$

such that

$$q^{l'}_{l'} = A^l_{l'} A^{l'}_k q^k_l = \delta^l_k \delta^k_l = \delta^l_l,$$

such that $\alpha(\{\mathbf{e}_i\}) = \alpha(\{\mathbf{e}_{i'}\})$. α is obviously a scalar. The argument in the matrix language should be that $\mathrm{trace}((A_,)^{-1}\, Q\, A_,) = \mathrm{trace}(Q^{'}_{,})$ for every invertible $n \times n$-matrix $A_,$.

∘ Consider a covariant 2-tensor ψ. Write

$$\psi(\{\mathbf{e}_i\}) = [q_{kl}] = Q.$$

Is the transformation $\beta : \mathrm{Bas}(V) \rightarrow \mathbb{R}$, defined by

$$\beta(\{\mathbf{e}_i\}) = \sum_{l=1}^{n} q_{ll} = \mathrm{trace}(Q),$$

a scalar? This represents the question if $\mathrm{trace}((A_,)^T\, Q\, A_,) = \mathrm{trace}(Q_{,,})$ for every invertible $n \times n$-matrix $A_,$? The answer is "no", so β is no scalar. □

Example(s): 2.9.3

- Given are the tensors q_{ij}, q^{ij}, q^i_j. Is the change of indices a tensor operation or in matrix language: "Is the transposition of a matrix a tensor operation?" For matrices with mixed indices this is not the case, but for matrices with lower or upper indices this is a tensor operation. The explanation will follow in matrix language.
  - ⋆ <u>Mixed indices</u>: Notate $Q = [q^i_j]$ and $Q^{'}_{,} = [q^{i'}_{j'}]$. Then holds $Q^{'}_{,} = (A_{,})^{-1}\, Q\, A_{,}$. Hereby follows that $(Q^{'}_{,})^T = (A_{,})^T\, Q^T\, (A_{,})^{-T}$. There follows that $(Q^{'}_{,})^T \neq (A_{,})^{-1}\, Q^T\, A_{,}$, in general, so the transposition of a matrix with mixed components is not a tensor operation.
  - ⋆ <u>Lower indices</u>: Notate $Q = [q^i_j]$ and $Q_{,,} = [q_{i'j'}]$. Then holds $Q_{,,} = (A_{,})^T\, Q\, A_{,}$. Hereby follows that $(Q_{,,})^T = (A_{,})^T\, Q^T\, A_{,}$. This is a tensor operation!
  - ⋆ <u>Upper indices</u>: Analogous as in the case of the lower indices.
- Given are the tensors q_{ij}, q^{ij}, q^i_j. Is the calculation of a determinant of a matrix is a tensor operation? To matrices with mixed indices it is a tensor operation, but to matrices with lower or upper indices it is not. This means that the calculation of a determinant of a matrix with mixed indices defines a scalar. Because $\det(Q^{'}_{,}) = \det((A_{,})^{-1}\, Q\, A_{,}) = \det(Q)$, but $\det(Q_{,,}) \neq \det((A_{,})^T\, Q\, A_{,})$, in general. □

Example(s): 2.9.4

- Let $n = 3$. Given are the contravariant 1-tensors x^i and y^j. Calculate the cross product $z^1 = x^2\, y^3 - x^3\, y^2$, $z^2 = x^3\, y^1 - x^1\, y^3$ and $z^3 = x^1\, y^2 - x^2\, y^1$. This is not a tensor operation, so z^k is not a contravariant 1-tensor. In other words: z^k is not a vector. To see that there is made use of the following calculation rule

$$\forall U, V \in \mathbb{R}^3\ \forall S \in \mathbb{R}^3_3, S \text{ invertible} \quad (SU) \times (SV) = \det(S)\, S^{-T}\, (U \times V). \tag{2.3}$$

If $Z^{'} = A^{'} Z$ then is z^k a vector. So the cross product is vector if $(A^{'} X) \times (A^{'} Y) = A^{'} (X \times Y)$. Because of the calculation rule 2.3 holds that $(A^{'} X) \times (A^{'} Y) = \det(A^{'})(A^{'})^{-T}(X \times Y)$, such that the cross product is not a tensor operation. But if the matrices $A^{'}$ are limited to the orthogonal matrices with determinant equal to 1, then the cross product is a tensor operation. If there is used an orthogonal matrix with determinant equal to -1, then there appears a minus sign after the basis transition. This phenomenon requires the physicists to call the cross product the mysterious title "axial vector". □

Example(s): 2.9.5

- Let x^i and y^j be contravariant 1-tensors. The arithmetic constructions $x^i y^j$ and $x^i y^j - y^j x^i$ deliver contravariant 2-tensors.
- If x^i, y_j, q_{ij}, q^i_j and q_{ij} are tensors then are $x^i q_{ij}$, $q^j_i y_j$, $q_{ij} q^j_k$, et cetera, tensors. These operations are to interpret geometrical as linear transformations on a covector, vector or as the composition of 2 linear transformations. □

Section 2.10 Symmetric and Antisymmetric Tensors

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$.

Definition 2.10.1 A permutation σ of order k, $k \in \mathbb{N}$, is a bijective transformation from $\{1, \cdots, k\}$ to $\{1, \cdots, k\}$. A permutation σ is called odd if σ is realised with an odd number of pairwise changes. A permutation σ is called even if σ is realised with an even number of pairwise changes. If a permutation is odd, there is written $\mathrm{sgn}(\sigma) = -1$. If a permutation is even, there is written $\mathrm{sgn}(\sigma) = 1$. The set of permutations of order k is notated by S_k. □

Comment(s): 2.10.1

- The number of elements out of S_k is equal to $k!$.

Definition 2.10.2 Let $\phi \in T_k(V) (= T^0_k(V))$. The tensor ϕ is called symmetric if for every number of k vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k \in V$ and for every $\sigma \in S_k$ holds that $\phi(\mathbf{v}_1, \cdots, \mathbf{v}_k) = \phi(\mathbf{v}_{\sigma(1)}, \cdots, \mathbf{v}_{\sigma(k)})$. The tensor ϕ is called antisymmetric if for every number of k vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k \in V$ and for every $\sigma \in S_k$ holds that $\phi(\mathbf{v}_1, \cdots, \mathbf{v}_k) = \mathrm{sgn}(\sigma)\, \phi(\mathbf{v}_{\sigma(1)}, \cdots, \mathbf{v}_{\sigma(k)})$. □

Comment(s): 2.10.2

- If $k > n$, then is every antisymmetric tensor equal to the tensor which adds 0 tot every elemenst of its domain.
- The change of an arbitrary pair of "input"-vectors has no influence to a symmetric tensor, it gives a factor -1 to an antisymmetric tensor.
- The sets of the symmetric and the antisymmetric k-tensors are subspaces of $T_k(V)$.

Notation(s):

- The Vector Space of the symmetric k-tensors is notated by $\bigvee^k(V)$.
- The Vector Space of the antisymmetric k-tensors is notated by $\bigwedge^k(V)$.
- The agreement is that $\bigvee^0(V) = \bigwedge^0(V) = \mathbb{R}$ and $\bigvee^1(V) = \bigwedge^1(V) = V^*$.

Definition 2.10.3 For every $\widehat{\mathbf{f}}, \widehat{\mathbf{g}} \in V^*$ the 2-tensor $\widehat{\mathbf{f}} \wedge \widehat{\mathbf{g}}$ on V is defined by

$$\left(\widehat{\mathbf{f}} \wedge \widehat{\mathbf{g}}\right)(\mathbf{x}, \mathbf{y}) = \det \begin{pmatrix} < \widehat{\mathbf{f}}, \mathbf{x} > & < \widehat{\mathbf{f}}, \mathbf{y} > \\ < \widehat{\mathbf{g}}, \mathbf{x} > & < \widehat{\mathbf{g}}, \mathbf{y} > \end{pmatrix}.$$

□

Lemma 2.10.1 The 2-tensor $\widehat{\mathbf{f}} \wedge \widehat{\mathbf{g}}$ is antisymmetric. □

Notice(s): 2.10.1

- $\widehat{\mathbf{f}} \wedge \widehat{\mathbf{g}} = -\widehat{\mathbf{g}} \wedge \widehat{\mathbf{f}}$, $\widehat{\mathbf{f}} \wedge \widehat{\mathbf{f}} = 0$ and $\left(\widehat{\mathbf{f}} + \lambda \widehat{\mathbf{g}}\right) \wedge \widehat{\mathbf{g}} = \widehat{\mathbf{f}} \wedge \widehat{\mathbf{g}}$ for all $\lambda \in \mathbb{R}$.
- For every basis $\{\mathbf{e}_i\}$ in V the set $\{\widehat{\mathbf{e}}^i \wedge \widehat{\mathbf{e}}^j \mid 1 \leq i < j \leq n\}$ is linear independent.

Definition 2.10.4 For every $\widehat{\mathbf{f}}_1, \cdots, \widehat{\mathbf{f}}_k \in V^*$ the k-tensor $\widehat{\mathbf{f}}_1 \wedge \cdots \wedge \widehat{\mathbf{f}}_k$ on V is defined by

$$\left(\widehat{\mathbf{f}}_1 \wedge \cdots \wedge \widehat{\mathbf{f}}_k\right)(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \det \begin{pmatrix} <\widehat{\mathbf{f}}_1, \mathbf{x}_1> & \cdots & <\widehat{\mathbf{f}}_1, \mathbf{x}_k> \\ \vdots & & \vdots \\ <\widehat{\mathbf{f}}_k, \mathbf{x}_1> & \cdots & <\widehat{\mathbf{f}}_k, \mathbf{x}_k> \end{pmatrix}.$$

□

Lemma 2.10.2 The k-tensor $\widehat{\mathbf{f}}_1 \wedge \cdots \wedge \widehat{\mathbf{f}}_k$ is antisymmetric. □

Notice(s): 2.10.2

- $\widehat{\mathbf{f}}_1 \wedge \cdots \wedge \widehat{\mathbf{f}}_k = 0$ if and only if the set $\{\widehat{\mathbf{f}}_1, \cdots, \widehat{\mathbf{f}}_k\}$ is linear dependent.

Lemma 2.10.3 For every basis $\{\mathbf{e}_i\}$ of V is the set

$$\{\widehat{\mathbf{e}}^{j_1} \wedge \cdots \wedge \widehat{\mathbf{e}}^{j_k} \,|\, 1 \leq j_1 < j_2 < \cdots < j_k \leq n\}$$

a basis of $\bigwedge^k(V)$. □

Clarification(s): 2.10.1

- Every antisymmetric k-tensor t on V can be written by

$$t = \sum_{1 \leq i_1 < \cdots < i_k \leq n} t_{i_1 \cdots i_k} \widehat{\mathbf{e}}^{i_1} \wedge \cdots \wedge \widehat{\mathbf{e}}^{i_k}, \tag{2.4}$$

with $t_{i_1 \cdots i_k} = t(\mathbf{e}_{i_1}, \cdots \mathbf{e}_{i_k})$.

Consequence(s):

- The dimension of $\bigwedge^k(V)$ is equal to $\binom{n}{k}$.

Definition 2.10.5 For every $\widehat{\mathbf{f}}, \widehat{\mathbf{g}} \in V^*$ the 2-tensor $\widehat{\mathbf{f}} \vee \widehat{\mathbf{g}} = \widehat{\mathbf{f}}\,\widehat{\mathbf{g}}$ on V is defined by

$$\left(\widehat{\mathbf{f}}\,\widehat{\mathbf{g}}\right)(\mathbf{x}, \mathbf{y}) = \mathrm{perm}\begin{pmatrix} < \widehat{\mathbf{f}}, \mathbf{x} > & < \widehat{\mathbf{f}}, \mathbf{y} > \\ < \widehat{\mathbf{g}}, \mathbf{x} > & < \widehat{\mathbf{g}}, \mathbf{y} > \end{pmatrix}.$$

□

Comment(s): 2.10.3

- In this definition is made use of the operator <u>perm</u>, which is called <u>permanent</u>. Perm adds a number to a matrix. The calculation is almost the same as the calculation of a determinant, the only difference is that there stays a plus sign for every form instead of alternately a plus or minus sign.

Lemma 2.10.4 The 2-tensor $\widehat{\mathbf{f}}\,\widehat{\mathbf{g}}$ is symmetric. □

Notice(s): 2.10.3

- $\widehat{\mathbf{f}}\,\widehat{\mathbf{g}} = \widehat{\mathbf{g}}\,\widehat{\mathbf{f}}$.
- $\widehat{\mathbf{f}}\,\widehat{\mathbf{g}} = 0 \Leftrightarrow \left(\widehat{\mathbf{f}} = 0 \text{ and/or } \widehat{\mathbf{g}} = 0\right)$.
- $\left(\widehat{\mathbf{f}} + \lambda\widehat{\mathbf{g}}\right) \vee \widehat{\mathbf{g}} = \widehat{\mathbf{f}} \vee \widehat{\mathbf{g}} + \lambda\widehat{\mathbf{g}} \vee \widehat{\mathbf{g}} = \widehat{\mathbf{f}}\,\widehat{\mathbf{g}} + \lambda\widehat{\mathbf{g}}\,\widehat{\mathbf{g}}$ for all $\lambda \in \mathbb{R}$.
- For every basis $\{\mathbf{e}_i\}$ of V is the set $\{\widehat{\mathbf{e}}^i\widehat{\mathbf{e}}^j \mid 1 \leq i \leq j \leq n\}$ linear independent.

Definition 2.10.6 For every $\widehat{\mathbf{f}}_1, \cdots, \widehat{\mathbf{f}}_k \in V^*$ the k-tensor $\widehat{\mathbf{f}}_1 \cdots \widehat{\mathbf{f}}_k$ on V is defined by

$$\left(\widehat{\mathbf{f}}_1 \cdots \widehat{\mathbf{f}}_k\right)(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \mathrm{perm}\begin{pmatrix} < \widehat{\mathbf{f}}_1, \mathbf{x}_1 > & \cdots & < \widehat{\mathbf{f}}_1, \mathbf{x}_k > \\ \vdots & & \vdots \\ < \widehat{\mathbf{f}}_k, \mathbf{x}_1 > & \cdots & < \widehat{\mathbf{f}}_k, \mathbf{x}_k > \end{pmatrix}.$$

□

Lemma 2.10.5 The k-tensor $\widehat{\mathbf{f}}_1 \cdots \widehat{\mathbf{f}}_k$ is symmetric. □

Notice(s): 2.10.4

- The order in $\widehat{\mathbf{f}_1} \cdots \widehat{\mathbf{f}_k}$ is not of importance, another order gives the same symmetric k-tensor.
- $\widehat{\mathbf{f}_1} \cdots \widehat{\mathbf{f}_k} = 0$ if and only if there exists an index j such that $\widehat{\mathbf{f}_k} = 0$.

Lemma 2.10.6 For every basis $\{\mathbf{e}_i\}$ of V is the set

$$\{\widehat{\mathbf{e}}^{j_1} \cdots \widehat{\mathbf{e}}^{j_k} \,|\, 1 \leq j_1 \leq \cdots \leq j_k \leq n\}$$

a basis of $\bigvee^k(V)$. □

Clarification(s): 2.10.2

- Every symmetric k-tensor τ on V can be written as

$$\tau = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \tau_{i_1 \cdots i_k} \frac{\widehat{\mathbf{e}}^{i_1} \cdots \widehat{\mathbf{e}}^{i_k}}{\mu_{i_1 \cdots i_k}},$$

with $\tau_{i_1 \cdots i_k} = \tau(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k})$ and $\mu_{i_1 \cdots i_k} = \left(\widehat{\mathbf{e}}^{i_1} \cdots \widehat{\mathbf{e}}^{i_k}\right)(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k})$. In this last expression the Einstein summation convention is not applicable!

Consequence(s):

- The dimension of $\bigvee^k(V)$ is equal to $\binom{n+k-1}{k}$.

Example(s): 2.10.1

- The m-th derivative of a sufficiently enough differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is a symmetric covariant m-tensor on $\mathbb{R}^n$. This tensor is noated by $\mathcal{D}^m j$ and the components are

$$\frac{\partial^m f(0)}{\partial(x^1)^{i_1} \cdots \partial(x^n)^{i_m}}$$

with $i_1 + \cdots + i_m = m$ wit respect to the basis $\{\frac{\partial}{\partial x^i}\}$. □

Definition 2.10.7 The transformation $\mathcal{S} : T_k(V) \rightarrow T_k(V)$ defined by

$$(\mathcal{S}\phi)(\mathbf{v}_1, \cdots, \mathbf{v}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \phi(\mathbf{v}_{\sigma_1}, \cdots, \mathbf{v}_{\sigma_k})$$

is called the symmetrizing transformation and the transformation $\mathcal{A} : T_k(V) \rightarrow T_k(V)$ defined by

$$(\mathcal{A}\phi)(\mathbf{v}_1, \cdots, \mathbf{v}_k) = \frac{1}{k!} \operatorname{sgn}(\sigma) \sum_{\sigma \in S_k} \phi(\mathbf{v}_{\sigma_1}, \cdots, \mathbf{v}_{\sigma_k})$$

is called the antisymmetrizing transformation. □

Notice(s): 2.10.5

- The transformations $\mathcal{S}$ and $\mathcal{A}$ are linear and they satisfy $\mathcal{S}^2 = \mathcal{S}$ and $\mathcal{A}^2 = \mathcal{A}$. These relations express that $\mathcal{A}$ and $\mathcal{A}$ are projections. The images are given by

$$\mathcal{S}(T_k(V)) = \bigvee^k(V) \text{ and } \mathcal{A}(T_k(V)) = \bigwedge^k(V).$$

- $\mathcal{A}\left(\widehat{\mathbf{e}}^{i_1} \otimes \cdots \otimes \widehat{\mathbf{e}}^{i_k}\right) = \frac{1}{k!}\widehat{\mathbf{e}}^{i_1} \wedge \cdots \wedge \widehat{\mathbf{e}}^{i_k}$.
- $\mathcal{S}\left(\widehat{\mathbf{e}}^{i_1} \otimes \cdots \otimes \widehat{\mathbf{e}}^{i_k}\right) = \frac{1}{k!}\widehat{\mathbf{e}}^{i_1} \cdots \widehat{\mathbf{e}}^{i_k}$.
- A 2-tensor can always be written as the sum of a symmetrical and an antisymmetrical 2-tensor. Consider the covariant components of a 2-tensor ϕ on V, $\phi_{ij} = \frac{1}{2}(\phi_{ij} + \phi_{ji}) + \frac{1}{2}(\phi_{ij} - \phi_{ji})$.
- For $k > 2$ then $\binom{n}{k} + \binom{n+k-1}{k} < n^k$, such that the Vector Spaces $\bigvee^k(V)$ and $\bigwedge^k(V)$ together don't span the space $T_k(V)$.

Definition 2.10.8 If $\eta \in \bigwedge^k(V)$ and $\zeta \in \bigwedge^l(V)$ then the tensor $\eta \wedge \zeta \in \bigwedge^{k+l}(V)$ is defined by

$$\eta \wedge \zeta = \frac{(k+l)!}{k!\,l!} \mathcal{A}(\eta \otimes \zeta).$$

□

Comment(s): 2.10.4

- If $k = l = 1$ then $\eta \wedge \zeta = \eta \otimes \zeta - \zeta \otimes \eta$, just conform Definition 2.10.3.
- If α is a scalar then holds $\alpha \wedge \eta = \alpha\eta$.

Theorem 2.10.1 For $\eta \in \bigwedge^k(V)$, $\zeta \in \bigwedge^l(V)$, $\theta \in \bigwedge^m(V)$ and $\omega \in \bigwedge^m(V)$ holds that

$$\begin{aligned} \eta \wedge \zeta &= (-1)^{(kl)}\, \zeta \wedge \eta \\ \eta \wedge (\zeta \wedge \theta) &= (\eta \wedge \zeta) \wedge \theta \\ (\eta + \omega) \wedge \zeta &= \eta \wedge \zeta + \omega \wedge \zeta. \end{aligned}$$

□

Clarification(s): 2.10.3

- The proof of the given theorem is omitted. The proof is a no small accounting and combinatorial issue, which can be found in (Abraham et al., 2001) ,Manifolds, $\cdots$, page 387.
- The practical calculations with the wedge product $\wedge$ are done following obvious rules. For instance if $k = 2$, $l = 1$ and $\eta = \alpha \widehat{\mathbf{u}} \wedge \widehat{\mathbf{w}} + \beta \widehat{\mathbf{v}} \wedge \widehat{\mathbf{x}}$, $\zeta = \gamma \widehat{\mathbf{x}} + \delta \widehat{\mathbf{z}}$, dan geldt $\eta \wedge \zeta = \alpha\gamma \widehat{\mathbf{u}} \wedge \widehat{\mathbf{w}} \wedge \widehat{\mathbf{x}} + \alpha\delta \widehat{\mathbf{u}} \wedge \widehat{\mathbf{w}} \wedge \widehat{\mathbf{z}} + \beta\delta \widehat{\mathbf{v}} \wedge \widehat{\mathbf{x}} \wedge \widehat{\mathbf{z}}$.

Example(s): 2.10.2

- Consider $\mathbb{R}^2$ with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, given by $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The associated dual basis is given by $\{\widehat{\mathbf{e}}^1, \widehat{\mathbf{e}}^2\}$. The following notations are here employed $\mathbf{e}_1 = \dfrac{\partial}{\partial x}, \mathbf{e}_2 = \dfrac{\partial}{\partial y}$ and $\widehat{\mathbf{e}}^1 = \mathrm{d}x, \widehat{\mathbf{e}}^2 = \mathrm{d}y$.

  The Vector Space $\bigwedge^1(\mathbb{R}^2) = (\mathbb{R}^2)^*$ is 2-dimensional and a basis of this space is given by $\{\mathrm{d}x, \mathrm{d}y\}$. Let $\widehat{\alpha}, \widehat{\beta} \in \bigwedge^1(\mathbb{R}^2)$ and expand these covectors to their covariant components with respet to the basis $\{\mathrm{d}x, \mathrm{d}y\}$. So $\widehat{\alpha} = \alpha_1 \mathrm{d}x + \alpha_2 \mathrm{d}y$, with $\alpha_1 = \widehat{\alpha}\dfrac{\partial}{\partial x}$ and $\alpha_2 = \widehat{\alpha}\dfrac{\partial}{\partial y}$. On the same way $\widehat{\beta} = \beta_1 \mathrm{d}x + \beta_2 \mathrm{d}y$ and rhere follows that

$$\begin{aligned} \widehat{\alpha} \wedge \widehat{\beta} &= (\alpha_1 \mathrm{d}x + \alpha_2 \mathrm{d}y) \wedge (\beta_1 \mathrm{d}x + \beta_2 \mathrm{d}y) \\ &= \alpha_1\beta_2 \mathrm{d}x \wedge \mathrm{d}y + \alpha_2\beta_1 \mathrm{d}y \wedge \mathrm{d}x = (\alpha_1\beta_2 - \alpha_2\beta_1)\mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

  Let $\mathbf{a} = \begin{pmatrix} a^1 \\ a^2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} \in \mathbb{R}^2$. The numbers a^1, a^2, b^1 and b^2 are the contravariant components of $\mathbf{a}$ and $\mathbf{b}$ with respect tot the basis $\{\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\}$. There holds that

$$(\mathrm{d}x \wedge \mathrm{d}y)(\mathbf{a}, \mathbf{b}) = <\mathrm{d}x, \mathbf{a}><\mathrm{d}y, \mathbf{b}> - <\mathrm{d}x, \mathbf{b}><\mathrm{d}y, \mathbf{a}> = \alpha_1\beta_2 - \alpha_2\beta_1.$$

  This number is the oriented surface of the parallelogram spanned by the vectors **a** and **b**.

  The Vector Space $\bigwedge^2(\mathbb{R}^2)$ is 1-dimensional and a basis is given by $\{\mathrm{d}x \wedge \mathrm{d}y\}$. □

Example(s): 2.10.3

- Consider $\mathbb{R}^3$ with basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ given by $\mathbf{e}_1 = \dfrac{\partial}{\partial x} = (1,0,0)^T, \mathbf{e}_2 = \dfrac{\partial}{\partial y} = (0,1,0)^T$ and $\mathbf{e}_3 = \dfrac{\partial}{\partial z} = (0,0,1)^T$. The corresponding dual basis is notated by $\{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z\}$.
  The basis of the 3-dimensional Vector Space $\bigwedge^1(\mathbb{R}^3)$ is $\{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z\}$. The basis of the 3-dimensional Vector Space $\bigwedge^2(\mathbb{R}^3)$ is $\{\mathrm{d}x \wedge \mathrm{d}y, \mathrm{d}x \wedge \mathrm{d}z, \mathrm{d}y \wedge \mathrm{d}z\}$, and the basis of the 1-dimensional Vector Space $\bigwedge^3(\mathbb{R}^3)$ is $\{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\}$.
  Let $\alpha, \beta \in \bigwedge^1(\mathbb{R}^3)$ then $\alpha = \alpha_1 \mathrm{d}x + \alpha_2 \mathrm{d}y + \alpha_3 \mathrm{d}z$ and $\beta = \beta_1 \mathrm{d}x + \beta_2 \mathrm{d}y + \beta_3 \mathrm{d}z$ and $\alpha \wedge \beta \in \bigwedge^2(\mathbb{R}^3)$. There holds that

$$\alpha \wedge \beta = (\alpha_1\beta_2 - \alpha_2\beta_1)\mathrm{d}x \wedge \mathrm{d}y + (\alpha_1\beta_3 - \alpha_3\beta_1)\mathrm{d}x \wedge \mathrm{d}z + (\alpha_2\beta_3 - \alpha_3\beta_2)\mathrm{d}y \wedge \mathrm{d}z.$$

  Let $\mathbf{a} = (a^1, a^2, a^3)^T$, $\mathbf{b} = (b^1, b^2, b^3)^T$, $\mathbf{c} = (c^1, c^2, c^3)^T \in \mathbb{R}^3$ then holds

$$(\mathrm{d}y \wedge \mathrm{d}z)(\mathbf{a}, \mathbf{b}) = a^2 b^3 - b^2 a^3.$$

  This number is the oriented surface of the projection on the y, z-plane of the parallelogram spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$. In addition holds that

$$\begin{aligned}
&(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \\
&(\mathrm{d}x \otimes \mathrm{d}y \otimes \mathrm{d}z)(\mathbf{a}, \mathbf{b}, \mathbf{c}) + (\mathrm{d}y \otimes \mathrm{d}z \otimes \mathrm{d}x)(\mathbf{a}, \mathbf{b}, \mathbf{c}) + \\
&(\mathrm{d}z \otimes \mathrm{d}x \otimes \mathrm{d}y)(\mathbf{a}, \mathbf{b}, \mathbf{c}) - (\mathrm{d}y \otimes \mathrm{d}x \otimes \mathrm{d}z)(\mathbf{a}, \mathbf{b}, \mathbf{c}) - \\
&(\mathrm{d}x \otimes \mathrm{d}z \otimes \mathrm{d}y)(\mathbf{a}, \mathbf{b}, \mathbf{c}) - (\mathrm{d}z \otimes \mathrm{d}y \otimes \mathrm{d}x)(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \\
&a^1 b^2 c^3 + a^2 b^3 c^1 + a^3 b^1 c^2 - a^2 b^1 c^3 - a^1 b^3 c^2 - a^3 b^2 c^1.
\end{aligned}$$

  This number is the oriented volume of the parallelepiped spanned by the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. □

Example(s): 2.10.4

- Consider $\mathbb{R}^4$ with basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ given by $\mathbf{e}_0 = \frac{\partial}{\partial t} = (1, 0, 0, 0)^T, \mathbf{e}_1 = \frac{\partial}{\partial x} = (0, 1, 0, 0)^T, \mathbf{e}_2 = \frac{\partial}{\partial y} = (0, 0, 1, 0)^T$ and $\mathbf{e}_3 = \frac{\partial}{\partial z} = (0, 0, 0, 1)^T$. The corresponding dual basis is notated by $\{\mathrm{d}t,\ \mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z\}$.
  The basis of the 4-dimensional Vector Space $\bigwedge^1(\mathbb{R}^4)$ is $\{\mathrm{d}t,\ \mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z\}$.
  The basis of the 6-dimensional Vector Space $\bigwedge^2(\mathbb{R}^4)$ is
  $\{\mathrm{d}t \wedge \mathrm{d}x,\ \mathrm{d}t \wedge \mathrm{d}y,\ \mathrm{d}t \wedge \mathrm{d}z,\ \mathrm{d}x \wedge \mathrm{d}y,\ \mathrm{d}x \wedge \mathrm{d}z,\ \mathrm{d}y \wedge \mathrm{d}z\}$.
  The basis of the 4-dimensional Vector Space $\bigwedge^3(\mathbb{R}^4)$ is
  $\{\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y,\ \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}z,\ \mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z,\ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\}$.
  The basis of the 1-dimensional Vector Space $\bigwedge^4(\mathbb{R}^4)$ is $\{\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\}$.
  Let $\alpha = \alpha_{01}\mathrm{d}t \wedge \mathrm{d}x + \alpha_{12}\mathrm{d}x \wedge \mathrm{d}y + \alpha_{13}\mathrm{d}x \wedge \mathrm{d}z \in \bigwedge^2(\mathbb{R}^4)$ and $\beta = \beta_0\mathrm{d}t + \beta_2\mathrm{d}y \in \bigwedge^1(\mathbb{R}^4)$ then $\alpha \wedge \beta \in \bigwedge^3(\mathbb{R}^4)$ and there holds that

  $$\alpha \wedge \beta = (\alpha_{01}\beta_2 + \alpha_{12}\beta 0)\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y + \alpha_{13}\beta_0\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}z - \alpha_{13}\beta_2\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

  Let $\gamma = \gamma_{23}\mathrm{d}y \wedge \mathrm{d}z \in \bigwedge^2(\mathbb{R}^4)$ then $\alpha \wedge \gamma \in \bigwedge^4(\mathbb{R}^4)$ and there holds that

  $$\alpha \wedge \gamma = \alpha_{01}\,\gamma_{23}\,\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

  Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^4$ and these vectors are expanded to their contravariant components with respect to the basis $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$. There holds that

  $$(\mathrm{d}t \wedge \mathrm{d}z)(\mathbf{a}, \mathbf{b}) = a^0 b^3 - b^0 a^3.$$

  This number is the oriented surface of the projection on the t, z-plane of the parallelogram spanned by **a** and **b**.

  $$(\mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z)(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \det\begin{pmatrix} a^0 & b^0 & c^0 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{pmatrix}$$

  is the oriented 3-dimensional volume of the projection on the t, y, z-hyperplane of the parallelepiped spanned by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. Further is

  $$(\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \det(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$$

  the 4-dimensional volume of the hyperparallelepiped spanned by
  $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$. □

Comment(s): 2.10.5

- Through the choice of a $\mu \in \bigwedge^n(V)$ is introduced an <u>oriented volume</u> on V. The number $\mu(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ gives the volume of the parallelepiped spanned by the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$. Because the Vector Space $\bigwedge^n(V)$ is one dimensional, every two choices of μ differ some multiplicative constant. If there is defined an inner product on V, it is customary to choose μ such that for orthonormal bases $\{\mathbf{e}_i\}$ on V holds that $\mu(\mathbf{e}_1, \cdots, \mathbf{e}_n) = \pm 1$. A basis with the plus-sign(minus-sign) is called positive(negative) oriented.

Section 2.11 Vector Spaces with a oriented volume

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$.
- An oriented volume μ on V.

Definition 2.11.1 Let $k \in \{0, \cdots, (n-1)\}$ and $\mathbf{a}_1, \cdots, \mathbf{a}_{(n-k)} \in V$, then $\mu \lrcorner \mathbf{a}_1 \lrcorner \cdots \lrcorner \mathbf{a}_{(n-k)} \in \bigwedge^k(V)$ is defined by

$$(\mu \lrcorner \mathbf{a}_1 \lrcorner \cdots \lrcorner \mathbf{a}_{(n-k)})(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \mu(\mathbf{a}_1, \cdots, \mathbf{a}_{(n-k)}, \mathbf{x}_1, \cdots, \mathbf{x}_k).$$

$\square$

Comment(s): 2.11.1

- If for some basis $\{\mathbf{e}_i\}$ on V holds that $\mu(\mathbf{e}_1, \cdots, \mathbf{e}_n) = 1$ then

$$(\mu \lrcorner \mathbf{a}_1 \lrcorner \cdots \lrcorner \mathbf{a}_{(n-k)})(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \det \begin{pmatrix} a^1_1 & \cdots & a^1_{(n-k)} & x^1_1 & \cdots & x^1_k \\ \vdots & & \vdots & \vdots & & \vdots \\ a^n_1 & \cdots & a^n_{(n-k)} & x^n_1 & \cdots & x^n_k \end{pmatrix},$$

because of the representation given in Clarification 2.4, where $a^i_j = < \widehat{\mathbf{e}}^i, \mathbf{a}_j >$, for $i = 1, \cdots, n$, and $j = 1, \cdots, (n-k)$, and $x^i_j = < \widehat{\mathbf{e}}^i, \mathbf{x}_j >$, for $i = 1, \cdots, n$, and $j = 1, \cdots, k$. Developping this determinant to the first $(n-k)$ columns, then becomes clear that $(\mu \lrcorner \mathbf{a}_1 \lrcorner \cdots \lrcorner \mathbf{a}_{(n-k)})(\mathbf{x}_1, \cdots, \mathbf{x}_k)$ is writable as a linear combination of $\binom{n}{k}$ $k \times k$ determinants

$$\det \begin{pmatrix} x^{i_1}_1 & \cdots & x^{i_1}_k \\ \vdots & & \vdots \\ x^{i_k}_1 & \cdots & x^{i_k}_k \end{pmatrix},$$

with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.
This result means that the antisymmetric k-tensor $\mu \lrcorner \mathbf{a}_1 \lrcorner \cdots \lrcorner \mathbf{a}_{(n-k)}$ is a linear combination of the $\binom{n}{k}$ k-tensors $\widehat{\mathbf{e}}^{i_1} \wedge \cdots \wedge \widehat{\mathbf{e}}^{i_k}$. This result was to expect because of the fact that $\{\widehat{\mathbf{e}}^{i_1} \wedge \cdots \wedge \widehat{\mathbf{e}}^{i_k}\}$ is a basis of $\bigwedge^k(V)$.

Example(s): 2.11.1

- Consider $\mathbb{R}^3$ with basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, given by $\mathbf{e}_1 = (1,0,0)^T, \mathbf{e}_2 = (0,1,0)^T$ and $\mathbf{e}_3 = (0,0,1)^T$. Define the volume μ on V by $\mu = \widehat{\mathbf{e}}^1 \wedge \widehat{\mathbf{e}}^2 \wedge \widehat{\mathbf{e}}^3$. Then holds that $\mu(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1$.
  Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ then $\mu \lrcorner \mathbf{a} \lrcorner \mathbf{b} \in \bigwedge^1(\mathbb{R}^3)$ and there holds that
  $$(\mu \lrcorner \mathbf{a} \lrcorner \mathbf{b})(\mathbf{x}) = \det\begin{pmatrix} a^1 & b^1 & x^1 \\ a^2 & b^2 & x^2 \\ a^3 & b^3 & x^3 \end{pmatrix} = (a^2b^3 - a^3b^2)\, x^1 + (a^3b^1 - a^1b^3)\, x^2 + (a^3b^1 - a^1b^3)\, x^3,$$
  such that
  $$\mu \lrcorner \mathbf{a} \lrcorner \mathbf{b} = (a^2b^3 - a^3b^2)\widehat{\mathbf{e}}^1 + (a^3b^1 - a^1b^3)\widehat{\mathbf{e}}^2 + (a^3b^1 - a^1b^3)\widehat{\mathbf{e}}^3.$$
  In addition $\mu \lrcorner \mathbf{a} \in \bigwedge^2(\mathbb{R}^3)$ and there holds that
  $$(\mu \lrcorner \mathbf{a})(\mathbf{x}, \mathbf{y}) = \det\begin{pmatrix} a^1 & x^1 & y^1 \\ a^2 & x^2 & y^2 \\ a^3 & x^3 & y^3 \end{pmatrix}$$
  $$= a^1 \det\begin{pmatrix} x^2 & y^2 \\ x^3 & y^3 \end{pmatrix} + a^2 \det\begin{pmatrix} x^3 & y^3 \\ x^1 & y^1 \end{pmatrix} + a^3 \det\begin{pmatrix} x^1 & y^1 \\ x^2 & y^2 \end{pmatrix},$$
  or
  $$\mu \lrcorner \mathbf{a} = a^1 \widehat{\mathbf{e}}^2 \wedge \widehat{\mathbf{e}}^3 + a^2 \widehat{\mathbf{e}}^3 \wedge \widehat{\mathbf{e}}^1 + a^3 \widehat{\mathbf{e}}^1 \wedge \widehat{\mathbf{e}}^2.$$

□

Notice(s): 2.11.1

- If for the basis $\{\mathbf{e}_i\}$ of V holds that $\mu(\mathbf{e}_1, \cdots, \mathbf{e}_n) = 1$, then holds that $\mu = \widehat{\mathbf{e}}^1 \wedge \cdots \wedge \widehat{\mathbf{e}}^n$.
  Moreover holds for every $k \in \{1, \cdots, (n-1)\}$,
  $$\mu \lrcorner \mathbf{e}^1 \lrcorner \cdots \lrcorner \mathbf{e}^k = \widehat{\mathbf{e}}^{(k+1)} \wedge \cdots \wedge \widehat{\mathbf{e}}^n.$$
  Furthermore holds that
  $$\mu \lrcorner \mathbf{e}_{i_1} \lrcorner \cdots \lrcorner \mathbf{e}_{i_k} = (-1)^{\nu} \widehat{\mathbf{e}}^{j_1} \wedge \cdots \wedge \widehat{\mathbf{e}}^{j_{(n-k)}}.$$
  The $j_1, \cdots, j_{(n-k)}$ are the indices which are left over and ν is the amount of permutations to get the indices $i_1, \cdots, i_k, j_1, \cdots, j_{(n-k)}$ in their natural order $1, 2, \cdots, n$.

Section 2.12 The Hodge Transformation

Starting Point(s):

- A n-dimensional Vector Space V over $\mathbb{R}$.
- An oriented volume μ on V such that for orthonormal bases $\{\mathbf{c}_i\}$ of V holds that $\mu(\mathbf{c}_1, \cdots, \mathbf{c}_n) = \pm 1$.
- A positive oriented basis $\{\mathbf{e}_i\}$.

Clarification(s): 2.12.1

- The startingpoint of an inner product means that the inner product is symmetric, see Def. 2.5.1 i. In this paragraph it is of importance.

Comment(s): 2.12.1

- With the help of the inner product there can be made a bijection between V and V^*. This bijection is notated by $\mathcal{G}$, see Theorem 2.5.1. For every $\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y} \in V$ there holds that

$$(\mathcal{G}\mathbf{a} \wedge \mathcal{G}\mathbf{b})(\mathbf{x}, \mathbf{y}) = \left(\widehat{\mathbf{a}} \wedge \widehat{\mathbf{b}}\right)(\mathbf{x}, \mathbf{y}) = \det\begin{pmatrix} (\mathbf{a}, \mathbf{x}) & (\mathbf{a}, \mathbf{y}) \\ (\mathbf{b}, \mathbf{x}) & (\mathbf{b}, \mathbf{y}) \end{pmatrix}$$

and for $\mathbf{a}_1, \cdots, \mathbf{a}_k, \mathbf{x}_1, \cdots, \mathbf{x}_k \in V$ there holds that

$$\begin{aligned} &(\mathcal{G}\mathbf{a}_1 \wedge \cdots \wedge \mathcal{G}\mathbf{a}_k)(\mathbf{x}_1, \cdots \mathbf{x}_k) = \\ &\left(\widehat{\mathbf{a}}_1 \wedge \cdots \wedge \widehat{\mathbf{a}}_k\right)(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \\ &\det\begin{pmatrix} (\mathbf{a}_1, \mathbf{x}_1) & \cdots & (\mathbf{a}_1, \mathbf{x}_k) \\ \vdots & & \vdots \\ (\mathbf{a}_k, \mathbf{x}_1) & \cdots & (\mathbf{a}_k, \mathbf{x}_k) \end{pmatrix}. \end{aligned}$$

- Because of the fact that $\binom{n}{k} = \binom{n}{n-k}$, there holds that $\dim\left(\bigwedge^k(V)\right) = \dim\left(\bigwedge^{(n-k)}(V)\right)$. Through the choice of the inner product and the volume it is apparently possible to define an isomorphism between $\bigwedge^k(V)$ and $\bigwedge^{(n-k)}(V)$.

Definition 2.12.1 Let $\mathbf{a}_1, \cdots, \mathbf{a}_k \in V$.
The Hodge transformation $* : \bigwedge^k \Rightarrow \bigwedge^{(n-k)}(V)$ is defined by

$$\begin{cases} k = 0 : & *1 = \mu, \quad \text{followed by linear expansion}, \\ 0 < j \leq n : & *(\widehat{\mathbf{a}}^1 \wedge \cdots \wedge \widehat{\mathbf{a}}^k) = \mu \lrcorner \mathbf{a}_1 \lrcorner \cdots \lrcorner \mathbf{a}_k, \quad \text{followed by linear expansion}. \end{cases}$$

□

Example(s): 2.12.1

- Consider $\mathbb{R} = \bigwedge^0$ and let $\alpha \in \mathbb{R}$ then $*\alpha = \alpha\,\mu \in \bigwedge^n$.
- Consider $\mathbb{R}^3$ and the normal inner product and volume then holds that

$$*\left(\widehat{\mathbf{e}}^1 \wedge \widehat{\mathbf{e}}^2\right) = \mu \lrcorner \mathbf{e}_1 \lrcorner \mathbf{e}_2 = \widehat{\mathbf{e}}^3.$$

□

Notice(s): 2.12.1

- Consider an orthonormal basis $\{\mathbf{e}_i\}$ of V, such that $\mu(\mathbf{e}_1, \cdots, \mathbf{e}_n) = 1$. There is noted that the belonging Gram matrix is a diagonal matrix, at the first p diagonalelements have value 1 and the rest have value (-1). The number p is equal to the signature of the inner product, see Theorem 2.7.1. There holds that:

$$*\left(\widehat{\mathbf{e}}^{i_1} \wedge \cdots \wedge \widehat{\mathbf{e}}^{i_k}\right) = (-1)^r *\left(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}\right) =$$
$$(-1)^r \mu \lrcorner \mathbf{e}_{i_1} \lrcorner \cdots \mathbf{e}_{i_k} =$$
$$(-1)^{(r+\nu)} \widehat{\mathbf{e}}^{j_1} \wedge \cdots \wedge \widehat{\mathbf{e}}^{j_{(n-k)}},$$

with r is the number of negative values in $\{(\mathbf{e}_{i_1}, \mathbf{e}_{i_1}), \cdots, (\mathbf{e}_{i_k}, \mathbf{e}_{i_k})\}$, see 2.2 and ν is the amount of permutations to get the indices $i_1, \cdots, i_k, j_1, \cdots, j_{(n-k)}$ in their natural order $1, 2, \cdots, n$.

Example(s): 2.12.2

- Consider $\mathbb{R}^2$. Define the inner product on $\mathbb{R}^2$ by

$$(X, Y) = x^1 y^1 + x^2 y^2.$$

Let $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ be the standard basis of $\mathbb{R}^2$ and notate the corresponding dual basis by $\{dx, dy\}$. The same notation as used in Example 2.10.2. Notice that the standard basis is orthonormal. Define the oriented volume μ on V by $\mu = dx \wedge dy$ and notice that $\mu(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 1$. The isomorphism $\mathcal{G}$ is given by

$$\mathcal{G} = dx \otimes dx + dy \otimes dy.$$

Let $\alpha \in \bigwedge^0(\mathbb{R}^2)$ then holds that

$$*\alpha = \alpha(*1) = \alpha\mu = \alpha\, dx \wedge dy \in \bigwedge^2(\mathbb{R}^2).$$

Let $\alpha \in \bigwedge^1(\mathbb{R}^2)$ then holds that

$$*\alpha = *(\alpha_1 dx + \alpha_2 dy) = \alpha_1 * dx + \alpha_2 * dy = \alpha_1 * dy - \alpha_2 * dx = \in \bigwedge^1(\mathbb{R}^2).$$

Obviously holds that $* * \alpha = -\alpha$ and $\alpha \perp *\alpha$.
Let $\alpha \in \bigwedge^2(\mathbb{R}^2)$ then holds that

$$*\alpha = *(\alpha_{12}\, d \wedge dy) = \alpha_{12} * (d \wedge dy) = \alpha_{12} \in \bigwedge^0(\mathbb{R}^2).$$

Example(s): 2.12.3

- Consider $\mathbb{R}^3$. The used notations are the same as in Example 2.10.3. Define the inner product by

$$\mathcal{G} = dx \otimes dx + dy \otimes dy + dz \otimes dz$$

and the oriented volume μ by $\mu = dx \wedge dy \wedge dz$. There holds that

$$\begin{array}{llll} & * \, dx = dy \wedge dz & * \, (dx \wedge dy) = dz & \\ * \, 1 = dx \wedge dy \wedge dz & * \, dy = -dx \wedge dz & * \, (dx \wedge dz) = -dy & * \, (dx \wedge dy \wedge dz) = 1 \\ & * \, dz = dx \wedge dy & * \, (dy \wedge dz) = dx & \end{array}$$

Obviously holds that $** = I$, a property of the Euclidean $\mathbb{R}^3$.
Let $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$ and $\beta = \beta_1 dx + \beta_2 dy + \beta_3 dz$ then holds that

$$*(\alpha \wedge \beta) = (\alpha_2\beta_3 - \alpha_3\beta_2)\, dx + (\alpha_3\beta_1 - \alpha_1\beta_3)\, dy + (\alpha_1\beta_2 - \alpha_2\beta_1)\, dz.$$

- Consider $\mathbb{R}^4$ and see Example 2.10.4.
Define the inner product by

$$\mathcal{G} = dt \otimes dt - dx \otimes dx - dy \otimes dy - dz \otimes dz$$

(the Minkowski inner product and the oriented volume μ by $\mu = dt \wedge dx \wedge dy \wedge dz$, then holds that

$$\begin{array}{lll} & & * \, (dt \wedge dx) = -dy \wedge dz \\ & * \, dt = dx \wedge dy \wedge dz & * \, (dt \wedge dy) = dx \wedge dz \\ * \, 1 = dt \wedge dx \wedge dy \wedge dz & * \, dx = dt \wedge dy \wedge dz & * \, (dt \wedge dz) = -dx \wedge dy \\ & * \, dy = -dt \wedge dx \wedge dz & * \, (dx \wedge dy) = dt \wedge dz \\ & * \, dz = dt \wedge dx \wedge dy & * \, (dx \wedge dz) = -dt \wedge dy \\ & & * \, (dy \wedge dz) = dt \wedge dx \end{array}$$

$$\begin{array}{ll} * \, (dt \wedge dx \wedge dy) = dz & \\ * \, (dt \wedge dx \wedge dz) = -dy & * \, (dt \wedge dx \wedge dy \wedge dz) = 1 \\ * \, (dt \wedge dy \wedge dz) = dx & \\ * \, (dx \wedge dy \wedge dz) = dt & \end{array}$$

□

Note that the inner product has signature $(+,-,-,-)$ or $(-,+,+,+)$. Which signature is used, is a matter of convention. But today the signature $(+,-,-,-)$ is very often used, becomes standard.

Section 2.13 Exercises

1. Let V be a n-dimensional vector space over $\mathbb{R}$ with three bases $\{\mathbf{e}_i\}$, $\{\{bfe_{i'}\}$ and $\{\{bfe_{i''}\}$.
   Let see that
$$A_j^{j'} A_{j'}^{j''} = A_j^{j''}.$$

2. Let V be a symplectic vector space. Prove that the dimension of V is even and that axiom (*ii*) of the inner product can never be satisfied.

3. Prove the inequality of Cauchy-Schwarz, see Lemma 2.5.1.

4. Prove the uniqueness of the signature of the inner product.

5. Prove the identification of V with V^{**}. Indication: define a suitable linear transformation of V to V^{**} and prove that this is an isomorphism.

Section 2.14 RRvH: Identification V and V^*

Notice(s): 2.14.1

- Let U, V and W be vector spaces over $\mathbb{R}$.
- Important to note is that $U \times V$ is the cartesian product of sets, not the direct product of vector spaces. There is no algebraic structure on $U \times V$. Expressions as $(x, y) + (z, v)$ and $\alpha\,(x, y)$ are meaningless.
- If V is a vector space, there are classes of functions from $V \times V$ to W. The linear maps $\mathcal{L}(V \times V) \to W$, where $V \times V$ is the direct product of vector spaces (sometimes notated by $V \boxplus V$) and the bilinear maps $\hom(V, V\,; W)$, where $V \times V$ is just the cartesian product of sets.
  The only map, that is linear and bilinear, is the zero map.

Comment(s): 2.14.1 About the identification of V with V^{**} and the identification of V with V^*.

- Identification of V with V^{**}.
  Let the linear transformation $\psi : V \mapsto V^{**}$ be defined by $(\psi(x))(\phi) = \phi(x)$ then $\psi(x) \in (V^*)^* = V^{**}$. If $(\psi(x))(\phi) = 0$ for every $\phi \in V^*$ then $\phi(x) = 0$ for every $\phi \in V^*$ and there follows that $x = 0$. So ψ is injective, together with $\dim V^{**} = \dim V^* = \dim V = n < \infty$ gives that ψ is a bijective map between V and V^{**}.
  Nowhere is used a "structure". Nowhere are used coordinates or something like an inner product. ψ is called a canonical or natural isomorphism.
- Identification of V with V^*.
  The sets V and V^* contain completely different objects.
  There is needed some "structure" to identify V with V^*.

Definition 2.14.1 Let V be a vector space, a bilinear form $B : V \times V \to \mathbb{R}$ is non-degenerate if

$$\begin{cases} B(x, y) = 0 \text{ for all } x \in V & \text{then } y = 0 \text{ and} \\ B(x, y) = 0 \text{ for all } y \in V & \text{then } x = 0 \end{cases}.$$

□

Lemma 2.14.1 If there is a non-degenerate bilinear form $B : V \times V \to \mathbb{R}$ then the spaces V and V^* are isomorphic. □

Proof The bilinear form $B: V \times V \rightarrow \mathbb{R}$ defines an isomorphism between V and V^* by the formula $\psi(x): v \rightarrow B(v, x)$ with $x \in V$. $B(v, x) \in \mathbb{R}$ for every $x \in V$.
To proof that $\psi(x)$ is linear and one-to-one. Linearity is not difficult, it follows out of the bilinearity of B. $\psi(x)$ is injective, because if $\psi(x)(v) = \psi(x)(w)$ then $B(v, x) = B(w, x)$. Out of the bilinearity of B follows that $B(v - w, x) = 0$ for all x. B is non-degenerate, so $v - w = 0$ and that means that $v = w$.
Because furthermore $\dim V^* = \dim V = n < \infty$, $\psi(x)$ is also surjective, so $\psi(x)$ is bijective. □

Let $\phi \in V^*$ and $x \in V$ and define the bilinear form $B(\phi, x) = \phi(x)$.
Be aware of the fact that if $B: V^* \times V \rightarrow \mathbb{R}$, then

$$\begin{cases} B(\phi, \cdot) \in V^* & \text{with } \phi \text{ fixed and} \\ B(\cdot, x) \in (V^*)^* \; (\neq V^*) & \text{with x fixed} \end{cases}.$$

Let $\mathcal{B}$ is a bilinear form on the cartesian product $V \times V$, so $\mathcal{B}: V \times V \rightarrow \mathbb{R}$ then

$$\begin{cases} \mathcal{B}(x, \cdot) \in V^* & \text{with x is fixed and} \\ \mathcal{B}(\cdot, y) \in V^* & \text{with y is fixed} \end{cases}.$$

Example(s): 2.14.1

- Suppose V is the space of real polynomials in the variable x with degree less than or equal 2. A basis of this space is for instance $\{1, x, x^2\}$, so $\mathbf{e}_1 = 1$, $\mathbf{e}_2 = x$ and $\mathbf{e}_3 = x^2$. A basis of the dual space is defined by the covectors $\{\widehat{\mathbf{e}}^1, \widehat{\mathbf{e}}^2, \widehat{\mathbf{e}}^3\}$ with $\widehat{\mathbf{e}}^i(\mathbf{e}_j) = \delta^i_j$, with $1 \leq i \leq 3$ and $1 \leq j \leq 3$. Covectors can be computed, for instance by $\widehat{\mathbf{e}}^i(p) = \alpha^i_1 \, p(-1) + \alpha^i_1 \, p(0) + \alpha^i_1 \, p(1)$. After some calculations, the result is that $\widehat{\mathbf{e}}^1(p) = 1\,p(0), \widehat{\mathbf{e}}^2(p) = -\frac{1}{2}\,p(-1) + \frac{1}{2}\,p(1), \widehat{\mathbf{e}}^3(p) = \frac{1}{2}\,p(-1) - 1\,p(0) + \frac{1}{2}\,p(1)$. The covectors are linear functions. An arbitrary covector $\widehat{f} \in V^*$ is given by $\widehat{f} = \beta_1\widehat{\mathbf{e}}^1 + \beta_2\widehat{\mathbf{e}}^2 + \beta_3\widehat{\mathbf{e}}^3$ and $\widehat{f}(\alpha^1\mathbf{e}_1 + \alpha^2\mathbf{e}_2 + \alpha^3\mathbf{e}_3) = \beta_1\alpha^1 + \beta_2\alpha^2 + \beta_3\alpha^3$. But if the basis of V changes, the covectors also change. If for instance the basis of V is $\{1, 1 + x, x + x^2\}$ than the basis of covectors becomes $\widehat{\mathbf{e}}^1(p) = 1\,p(-1), \widehat{\mathbf{e}}^2(p) = -1\,p(-1) + 1\,p(0), \widehat{\mathbf{e}}^3(p) = \frac{1}{2}\,p(-1) - 1\,p(0) + \frac{1}{2}\,p(1)$.

Section 2.15 RRvH: Summary

In this summary is given an overview of the different operations, such as

$$< \cdot, \cdot >, \quad \otimes, \quad \wedge, \quad \vee, \quad \lrcorner\,, \quad \text{and} \quad * .$$

- First of all $< \cdot, \cdot >: V^* \times V \rightarrow \mathbb{R}$, the Kronecker tensor, see Section 2.3:

  $$< \widehat{u}, x > = \widehat{u}(x).$$

  If x is fixed, then it becomes a linear function on V^*, so a contravariant 1-tensor, if $\widehat{u}$ is fixed, then a linear function on V, so a covariant 1-tensor.
  With this Kronecker tensor are built all other kind of tensors.

- The $\binom{r}{s}$-tensors, see Section 2.8.10:

  $$\left(\mathbf{a} \otimes \mathbf{b} \otimes \cdots \otimes \mathbf{d} \otimes \widehat{\mathbf{p}} \otimes \widehat{\mathbf{q}} \otimes \cdots \otimes \widehat{\mathbf{u}}\right)(\underbrace{\widehat{\mathbf{v}}, \widehat{\mathbf{w}}, \cdots, \widehat{\mathbf{z}}}_{r \text{ covectors}}, \underbrace{\mathbf{f}, \mathbf{g}, \cdots, \mathbf{k}}_{s \text{ vectors}}) =$$

  $$< \widehat{\mathbf{v}}, \mathbf{a} > \cdots < \widehat{\mathbf{z}}, \mathbf{d} >< \widehat{\mathbf{p}}, \mathbf{f} > \cdots < \widehat{\mathbf{u}}, \mathbf{k} > \in T^r_s(V),$$

  with r vectors $\mathbf{a}, \mathbf{b}, \cdots, \mathbf{d} \in V$ and s covectors $\widehat{\mathbf{p}}, \widehat{\mathbf{q}}, \cdots, \widehat{\mathbf{u}} \in V^*$ are given.
  The $\binom{r}{s}$-tensor becomes a multiplication of $(r + s)$ real numbers.

- The k-tensor, see Section 2.10.4, can be seen as a construction where the $\binom{r}{s}$-tensors are used,

  $$\left(\widehat{\mathbf{f}}_1 \wedge \cdots \wedge \widehat{\mathbf{f}}_k\right)(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \det \begin{pmatrix} < \widehat{\mathbf{f}}_1, \mathbf{x}_1 > & \cdots & < \widehat{\mathbf{f}}_1, \mathbf{x}_k > \\ \vdots & & \vdots \\ < \widehat{\mathbf{f}}_k, \mathbf{x}_1 > & \cdots & < \widehat{\mathbf{f}}_k, \mathbf{x}_k > \end{pmatrix} \in \bigwedge\nolimits^k(V),$$

  with $\widehat{\mathbf{f}}_1, \cdots, \widehat{\mathbf{f}}_k \in V^*$.
  This tensor is antisymmetric.

- Another k-tensor, see Section 2.10.6 can also be seen as a construction where the $\binom{r}{s}$-tensors are used,

  $$\left(\widehat{\mathbf{f}}_1 \vee \cdots \vee \widehat{\mathbf{f}}_k\right)(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \operatorname{perm} \begin{pmatrix} < \widehat{\mathbf{f}}_1, \mathbf{x}_1 > & \cdots & < \widehat{\mathbf{f}}_1, \mathbf{x}_k > \\ \vdots & & \vdots \\ < \widehat{\mathbf{f}}_k, \mathbf{x}_1 > & \cdots & < \widehat{\mathbf{f}}_k, \mathbf{x}_k > \end{pmatrix} \in \bigvee\nolimits^k(V),$$

  with $\widehat{\mathbf{f}}_1, \cdots, \widehat{\mathbf{f}}_k \in V^*$. For the calculation of perm, see Comment 2.10.3. Another notation for this tensor is $\widehat{\mathbf{f}}_1 \cdots \widehat{\mathbf{f}}_k \; (= \widehat{\mathbf{f}}_1 \vee \cdots \vee \widehat{\mathbf{f}}_k)$.
  This tensor is symmetric.

- If μ is an oriented volume, see Comment 2.10.5, then

$$(\mu \lrcorner \mathbf{a}_1 \lrcorner \cdots \lrcorner \mathbf{a}_{(n-k)})(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \mu(\mathbf{a}_1, \cdots, \mathbf{a}_{(n-k)}, \mathbf{x}_1, \cdots, \mathbf{x}_k),$$

  see Definiton 2.11.1.
  If $\mu(\mathbf{e}_1, \cdots, \mathbf{e}_n) = 1$ then

$$(\mu \lrcorner \mathbf{a}_1 \lrcorner \cdots \lrcorner \mathbf{a}_{(n-k)})(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \det \begin{pmatrix} a^1_1 & \cdots & a^1_{(n-k)} & x^1_1 & \cdots & x^1_k \\ \vdots & & \vdots & \vdots & & \vdots \\ a^n_1 & \cdots & a^n_{(n-k)} & x^n_1 & \cdots & x^n_k \end{pmatrix},$$

  with $a^i_j = < \widehat{e}^i, a_j >$, for $i = 1, \cdots, n$, and $j = 1, \cdots, (n-k)$, and $x^i_j = < \widehat{e}^i, x_j >$, for $i = 1, \cdots, n$, and $j = 1, \cdots, k$, see Comment 2.11.1. This tensor is a linear combination of k-tensors and antisymmetric.

- The Hodge transformation $* : \bigwedge^k \Rightarrow \bigwedge^{(n-k)}(V)$ is defined by

$$\begin{cases} k = 0: & *1 = \mu, \quad \text{followed by linear expansion,} \\ 0 < j \leq n: & *(\widehat{\mathbf{a}}^1 \wedge \cdots \wedge \widehat{\mathbf{a}}^k) = \mu \lrcorner \mathbf{a}_1 \lrcorner \cdots \lrcorner \mathbf{a}_k, \quad \text{followed by linear expansion,} \end{cases}$$

  see Definition 2.12.1.

Section 2.16 RRvH: The four faces of bilinear maps

For the bijective linear transformation $G : V \to V^*$, see Theorem 2.5.1.

$$V \times V \xrightarrow{M} \mathbb{R}$$
$$(v, w) \longmapsto v' M w$$
$$M_{ij} = M(b_i, b_j)$$
$$M \in V^* \otimes V^*$$
$$M = M_{st}\, \beta^s \otimes \beta^t$$

$$V^* \times V \xrightarrow{M} \mathbb{R}$$
$$(f, w) \longmapsto f M w$$
$$M^i{}_j = M(\beta^i, b_j)$$
$$M \in V \otimes V^*$$
$$M = M^s{}_t\, b_s \otimes \beta^t$$
$$M^s{}_t = g^{su} M_{ut}$$
$$G^{-1} M$$

$$V \times V^* \xrightarrow{M} \mathbb{R}$$
$$(v, f) \longmapsto v' M f'$$
$$M_i^{\,j} = M(b_i, \beta^j)$$
$$M \in V^* \otimes V$$
$$M = M_s^{\,t}\, \beta^s \otimes b_t$$
$$M_s^{\,t} = M_{su}\, g^{ut}$$
$$M G^{-1}$$

$$V^* \times V^* \xrightarrow{M} \mathbb{R}$$
$$(f, g) \longmapsto f M g'$$
$$M^{ij} = M(\beta^i, \beta^j)$$
$$M \in V \otimes V$$
$$M = M^{st}\, b_s \otimes b_t$$
$$M^{st} = g^{su} M_{uv}\, g^{vt}$$
$$G^{-1} M G^{-1}$$

Chapter 3 Tensor Fields on $\mathbb{R}^n$

Section 3.1 Curvilinear Coordinates and Tangent Spaces

In this chapter are considered scalar fields, vector fields and the more general tensor fields on open subsets of $\mathbb{R}^n$. In the foregoing chapter is introduced the Vector Space $\mathbb{R}^n$ as the set of all real columns of length n with the usual definitions of addition and scalar multiplication. Elements of $\mathbb{R}^n$, which are also called points, are notated by X and the standard basis of $\mathbb{R}^n$ is notated by $\{E_i\}$, with

$$E_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T,$$

with the 1 at the i-th position. Every $X \in \mathbb{R}^n$ can be written as $X = x^i E_i$.

Definition 3.1.1 Let Ω be an open subset of $\mathbb{R}^n$. A system of n real-valued functions $\{f^i(X)\}$, defined on Ω, is called a (curvilinear) coordinate system for Ω, if the following conditions are satisfied:

- The map $\mathbf{f} = (f^1, \cdots, f^n)^T$ of Ω to $\mathbb{R}^n$ is injective. The following notation is used $u^i = f^i(x^i E_i)$. Notice that the functions f^i are functions of the variables x^i.
- The set $U = f(\Omega)$ is an open subset of $\mathbb{R}^n$.
- The map $\mathbf{f}$ is differentiable at every point $X \in \Omega$ and there holds also that $\det\left[\dfrac{\partial f^i}{\partial x^j}(X)\right] \neq 0$ for every $X \in \Omega$.

□

The map $\mathbf{f}$ is also called a chart map. The inverse map $\mathbf{f}^{\leftarrow} : U \to \Omega$ is called a parametrization of Ω. The variables x^j are functions of the variables u^i. If there is notated $\mathbf{f}^{\leftarrow} = (g^1, \cdots, g^n)$ then holds that $x^j = g^j(u^i)$. The chart map and the parametrization are often not to describe by simple functions. The inverse function theorem tells that $\mathbf{f}^{\leftarrow}$ is differentiable in every point of U and also that

$$\frac{\partial f^i}{\partial x^j}(x^1, \cdots, x^n) \frac{\partial g^j}{\partial u^k}(u^1, \cdots, u^n) = \delta^i_k,$$

with $u^l = f^l(x^1, \cdots, x^n)$. In corresponding points the matrices $\left[\dfrac{\partial f^i}{\partial x^j}\right]$ and $\left[\dfrac{\partial g^l}{\partial u^k}\right]$ are the inverse of each other. The curves, which are described by the equations $f^i(x^1, \cdots, x^n) = C$, with C a constant, are called curvilinear coordinates belonging to the coordinate curves.

Example(s): 3.1.1

- Let $\Omega = \mathbb{R}^n$. Cartesian coordinates are defined by $u^i = x^i$. There holds that $\det\left[\frac{\partial u^i}{\partial x^j}(X)\right] = \det\left[\delta^i_j\right] = 1$.
- Let $\Omega = \mathbb{R}^n$. Furthermore $B = \left[b^i\right] \in \mathbb{R}^n$ and $L = \left[L^i_j\right] \in \mathbb{R}^n_n$ with $\det L \neq 0$. General affine coordinates are defined by $u^i = b^i + L^i_j x^j$. There holds that $\det\left[\frac{\partial u^i}{\partial x^j}\right] = \det L \neq 0$. The parametrization is given by

$$x^j = (L^{-1})^j_k u^k - (L^{-1})^j_k b^k.$$

- Let $\Omega = \mathbb{R}^2 \backslash \{(x, 0) \mid x \in [0, \infty)\}$ and $U = (0, \infty) \times (0, 2\pi) \in \mathbb{R}^2$. Polar coordinates are defined by the parametrization

$$x = r\cos\phi,\ y = r\sin\phi, \text{ with } x = x^1,\ y = x^2,\ r = u^1 \text{ and } \phi = u^2.$$

With some effort, the corresponding chart map is to calculate. There holds that

$$r(x, y) = \sqrt{x^2 + y^2},$$

$$\phi(x, y) = \begin{cases} \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) & y \geq 0,\ x \neq 0 \\ 2\pi - \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) & y \leq 0,\ x \neq 0. \end{cases}$$

It is easy to examine that $\Omega = \mathbb{R}^2 \backslash \{(x, 0) \mid x \in (-\infty, 0]\}$ and $U = (0, \infty) \times (-\pi, \pi)$ would be also a good choice. □

The subject of study in this chapter is tensor fields on $\mathbb{R}^n$. Intuitive it means that at every point X of $\mathbb{R}^n$ (or of some open subset of it) there is added a tensor out of some Tensor Space, belonging to that point X. The "starting" Vector Space, which is added to every point X, is a copy of $\mathbb{R}^n$. To distinguish all these copies of $\mathbb{R}^n$, which belong to the point X, they are notated by $T_X(\mathbb{R}^n)$. Such a copy is called the Tangent Space in X. Let Ω be an open subset of $\mathbb{R}^n$. All these Tangent Spaces, which belong to the points $X \in \Omega$, are joined together to the so-called tangent bundle of Ω:

$$T(\Omega) = \bigcup_{X \in \Omega} T_X(\mathbb{R}^n) = \Omega \times \mathbb{R}^n = \{(X, \mathbf{x}) \mid X \in \Omega, \mathbf{x} \in \mathbb{R}^n\}.$$

The origins of all the Tangent Spaces $T_X(\mathbb{R}^n)$, with $X \in \Omega$, form together against the open subset Ω.

Let $x^k = g^k(u^1, \cdots, u^n)$ be a parametrization of Ω. The vectors $\mathbf{c}_i = \frac{\partial X}{\partial u^i}$ are tangent to the coordinate curves in X. So there is formed, on a natural way, a with the curvilinear coordinate x^k corresponding basis of $T_X(\mathbb{R}^n)$. Notice that if x^k are Cartesian coordinates then $\mathbf{c}_i = E_i$. This is in certain sense a copy of the standard basis of $\mathbb{R}^n$ shifted parallel to X.

The kernel index notation is used just as in the foregoing chapter. In stead of $u^i = u^i(x^1, \cdots, x^n)$ is written $x^{i'} = x^{i'}(x^1, \cdots, x^n) = x^{i'}(x^i)$ and analogous $x^i = x^i(x^{i'})$.

The matrix $\left[\frac{\partial x^{i'}}{\partial x^i}(x^k) \right]$ is invertible in every point X, because the determinant is supposed to be not equal to zero. Before is already noticed that the inverse in the point X is given by $\left[\frac{\partial x^{i}}{\partial x^{i'}}(x^{k'}) \right]$ with $x^{k'} = x^{k'}(x^k)$. Differentiation of $x^{i'} = x^{i'}(x^i)$ to $x^{j'}$ leads to

$$\delta^{i'}_{j'} = \frac{\partial x^{i'}}{\partial x^i}(x^k) \frac{\partial x^i}{\partial x^{i'}}(x^{k'}(x^k)).$$

The basis of $T_X(\mathbb{R}^n)$, associate with the coordinates x^i is notated by $\{\frac{\partial X}{\partial x^i}\}$ or shorter with $\{\frac{\partial}{\partial x^i}\}$. If there is a transition to other coordinates $x^{i'}$ there holds that

$$\frac{\partial}{\partial x^{i'}} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i},$$

such that the transition matrix of the basis $\{\frac{\partial}{\partial x^i}\}$ to the basis $\{\frac{\partial}{\partial x^{i'}}\}$ is equal to the matrix $\left[\frac{\partial x^i}{\partial x^{i'}} \right]$. Consequently the transition matrix of the basis $\{\frac{\partial}{\partial x^{i'}}\}$ to the basis $\{\frac{\partial}{\partial x^i}\}$ is given by the matrix $\left[\frac{\partial x^{i'}}{\partial x^i} \right]$.

In Chapter 2 is still spoken about a general Vector Space V. In the remaining lecture notes the Tangent Space $T_X(\mathbb{R}^n)$ plays at every point $X \in \mathbb{R}^n$ the rule of this general Vector Space V. At every point $X \in \mathbb{R}^n$ is added besides the Tangent Space $T_X(\mathbb{R}^n)$ also the Cotangent Space $T^*_X(\mathbb{R}^n) = (T_X(\mathbb{R}^n))^*$ and more general the Vector Space $T_{X}{}^r_s(\mathbb{R}^n)$ of the tensors which are covariant of the order s and contravariant of the order r. But it also possible to add subspaces such as the spaces of the symmetric or antisymmetric tensors, notated by $\bigvee_X(\mathbb{R}^n)$, respectively $\bigwedge_X(\mathbb{R}^n)$ at X.
To every basis of $T_X(\mathbb{R}^n)$ belongs also a dual basis of $T^*_X(\mathbb{R}^n)$. This dual basis, associated with the coordinates x^i, is notated by $\{\mathrm{d}x^i\}$. The dual basis $\{\mathrm{d}x^{i'}\}$ belonging to the coordinates $x^{i'}$ are founded with the matrix $\left[\frac{\partial x^{i'}}{\partial x^i} \right]$. There holds that

$$\mathrm{d}x^{i'} = \frac{\partial x^{i'}}{\partial x^i} \mathrm{d}x^i,$$

which follows out of Lemma 2.6.1. Be aware of the fact that the recipoke and dual bases are in accordance with each other after the choice of an inner product, see the conlusion at the end of Section 2.6 ◂. The result agrees with the folklore of the infinitesimal calculus!

Section 3.2 Definition of Tensor Fields on $\mathbb{R}^n$

Definition 3.2.1 A scalar field or $\binom{0}{0}$-tensor field φ on $\mathbb{R}^n$ is a map of $\mathbb{R}^n$ to $\mathbb{R}$. At every point $X \in \mathbb{R}^n$ there is added the number $\varphi(X)$. □

Definition 3.2.2 A vector field, contravariant vector field or $\binom{1}{0}$-tensor field $\mathbf{a}$ on $\mathbb{R}^n$ is a map of $\mathbb{R}^n$ to $\bigcup_{X \in \mathbb{R}^n} T_X(\mathbb{R}^n)$. At every point $X \in \mathbb{R}^n$ there is added a vector $\mathbf{a}(X)$ element out of the corresponding Tangent Space $T_X(\mathbb{R}^n)$. □

There belongs to a vector field $\mathbf{a}$ on $\mathbb{R}^n$, n functions a^i on $\mathbb{R}^n$, such that $\mathbf{a}(X) = a^i(x^k)\dfrac{\partial}{\partial x^i}$. In other (curvilinear) coordinates $x^{i'}$ is written $\mathbf{a}(x^k(x^{k'})) = a^{i'}(x^{k'})\dfrac{\partial}{\partial x^{i'}}$ and there holds that

$$a^{i'}(x^{k'}) = \frac{\partial x^{i'}}{\partial x^i}(x^k(x^{k'}))\, a^i(x^k(x^{k'})),$$

which is briefly written as $a^{i'} = \dfrac{\partial x^{i'}}{\partial x^i}\, a^i$.

Definition 3.2.3 A covector field, covariant vector field or $\binom{0}{1}$-tensor field α on $\mathbb{R}^n$ is a map of $\mathbb{R}^n$ to $\bigcup_{X \in \mathbb{R}^n} T^*_X(\mathbb{R}^n)$. At every point $X \in \mathbb{R}^n$ there is added an element $\alpha(X)$ out of the dual space $T^*_X(\mathbb{R}^n)$ of $T_X(\mathbb{R}^n)$. □

There belongs to a covector field α on $\mathbb{R}^n$, n functions α_i on $\mathbb{R}^n$, such that $\alpha(X) = \alpha_i(x^k)\mathrm{d}x^i$. In other (curvilinear) coordinates $x^{i'}$ is written $\alpha(x^k(x^{k'})) = \alpha_{i'}(x^{k'})\mathrm{d}x^{i'}$ and there holds that

$$\alpha_{i'}(x^{k'}) = \frac{\partial x^i}{\partial x^{i'}}(x^{k'})\, \alpha_i(x^k(x^{k'})),$$

which is briefly written as $\alpha_{i'} = \dfrac{\partial x^i}{\partial x^{i'}}\, \alpha_i$.

Definition 3.2.4 A $\binom{r}{s}$-tensor field on $\mathbb{R}^n$ is a map Φ of $\mathbb{R}^n$ to $\bigcup_{X \in \mathbb{R}^n} T_X{}^r_s(\mathbb{R}^n)$. At every point $X \in \mathbb{R}^n$ there is added a $\binom{r}{s}$-tensor $\Phi(X)$ an element out of $T_X{}^r_s(\mathbb{R}^n)$. □

There belongs to a $\binom{r}{s}$-tensor field on $\mathbb{R}^n$, $n^{(r+s)}$ functions $\Phi^{i_1 \cdots i_r}_{j_1 \cdots j_s}$, such that

$$\Phi(X) \;=\; \Phi^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x^k)\, \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}.$$

In other (curvilinear) coordinates $x^{i'}$ is written

$$\Phi(x^k(x^{k'})) \;=\; \Phi^{i'_1 \cdots i'_r}_{j'_1 \cdots j'_s}(x^{k'})\, \frac{\partial}{\partial x^{i'_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i'_r}} \otimes dx^{j'_1} \otimes \cdots \otimes dx^{j'_s},$$

and there holds that

$$\Phi^{i'_1 \cdots i'_r}_{j'_1 \cdots j'_s}(x^{k'}) \;=\; \frac{\partial x^{i'_1}}{\partial x^{i_1}}(x^k(x^{k'})) \cdots \frac{\partial x^{i'_r}}{\partial x^{i_r}}(x^k(x^{k'}))\, \frac{\partial x^{j_1}}{\partial x^{j'_1}}(x^{k'}) \cdots \frac{\partial x^{j_s}}{\partial x^{j'_s}}(x^{k'})\, \Phi^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x^k(x^{k'})),$$

which is briefly written as

$$\Phi^{i'_1 \cdots i'_r}_{j'_1 \cdots j'_s} \;=\; \frac{\partial x^{i'_1}}{\partial x^{i_1}}(x^k \cdots \frac{\partial x^{i'_r}}{\partial x^{i_r}}(x^k\, \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_s}}{\partial x^{j'_s}}\, \Phi^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x^k(x^{k'})).$$

Definition 3.2.5 A differential form of degree k or k-form θ on $\mathbb{R}^n$ is a map of $\mathbb{R}^n$ to $\bigcup_{X \in \mathbb{R}^n} \bigwedge_X{}^k(\mathbb{R}^n)$. At every point $X \in \mathbb{R}^n$ there is added an antisymmetric k-tensor $\theta(X)$ out of $\bigwedge_X{}^k(\mathbb{R}^n)$. □

A 0-form is a scalar field and an 1-form is a covector field. In fact every k-form is a $\binom{0}{k}$-tensor field, see Definition 3.2.4. This class of tensor vector fields is important. That is reason there is paid extra attention to these tensor fields.
To a k-form θ on $\mathbb{R}^n$ belong $\binom{n}{k}$ functions $\theta_{i_1 \cdots i_k}$, for $1 \leq i_1 \cdots i_k \leq n$, on $\mathbb{R}^n$ such that

$$\theta(X) \;=\; \sum_{1 \leq i_1 < \cdots < i_k \leq n} \theta_{i_1 \cdots i_k}(x^l)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{3.1}$$

(Compare this with the representation 2.4 in Section 2.10.)

Lemma 3.2.1 If in other (curvilinear) coordinates $x^{i'}$ is written

$$\theta(x^l(x^{l'})) \;=\; \sum_{1 \leq i'_1 < \cdots < i'_k \leq n} \theta_{i'_1 \cdots i'_k}(x^{l'})\, dx^{i'_1} \wedge \cdots \wedge dx^{i'_k}, \tag{3.2}$$

there holds that

$$\theta_{i'_1 \cdots i'_k}(x^{l'}) \;=\; \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathfrak{J}^{\,i_1 \cdots i_k}_{\,i'_1 \cdots i'_k}(x^{l'})\, \theta_{i_1 \cdots i_k}(x^l(x^{l'}))\,, \tag{3.3}$$

with

$$\mathfrak{J}^{\,i_1 \cdots i_k}_{\,i'_1 \cdots i'_k} \;=\; \frac{\partial(x^{i_1}, \cdots, x^{i_k})}{\partial(x^{i'_1}, \cdots, x^{i'_k})}.$$

□

Proof Consider the representation 3.1 and notice that

$$\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = \frac{\partial x^{i_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{i_k}}{\partial x^{j'_k}} \, \mathrm{d}x^{j'_1} \wedge \cdots \wedge \mathrm{d}x^{j'_k}. \tag{3.4}$$

The terms in the summation of the right part of 3.4 are not equal to zero if the indices j'_p for $p = 1, \cdots, k$ are not equal. Choose a fixed, ordered collection of indices $i'_1, \cdots, i'_k$ with $1 \leq i'_1 < \cdots < i'_k \leq n$. Choose now the terms in the summation of the right side of 3.4 such that the unordered collection $j'_1, \cdots, j'_k$ is exactly the collection $i'_1, \cdots, i'_k$. Note that there are $k!$ possibilities. To every unordered collection $j'_1, \cdots, j'_k$ there is exactly one $\sigma \in S_k$ such that $j'_p = i'_{\sigma(p)}$, for $p = 1, \cdots, k$. Out all of this follows

$$\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = \sum_{1 \leq i'_1 < \cdots < i'_k \leq n} \sum_{\sigma \in S_k} \frac{\partial x^{i_1}}{\partial x^{i'_{\sigma(1)}}} \cdots \frac{\partial x^{i_k}}{\partial x^{j'_{\sigma(k)}}} \, \mathrm{d}x^{i'_{\sigma(1)}} \wedge \cdots \wedge \mathrm{d}x^{i'_{\sigma(k)}}.$$

To put the term $\mathrm{d}x^{i'_{\sigma(1)}} \wedge \cdots \wedge \mathrm{d}x^{i'_{\sigma(k)}}$ into the order $\mathrm{d}x^{i'_1} \wedge \cdots \wedge \mathrm{d}x^{i'_k}$ has to be corrected with a factor $\mathrm{sgn}(\sigma)$, the factor obtained by the order of the permutation. So

$$\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = \sum_{1 \leq i'_1 < \cdots < i'_k \leq n} \left(\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \frac{\partial x^{i_1}}{\partial x^{i'_{\sigma(1)}}} \cdots \frac{\partial x^{i_k}}{\partial x^{j'_{\sigma(k)}}} \right) \mathrm{d}x^{i'_1} \wedge \cdots \wedge \mathrm{d}x^{i'_k}.$$

In the term between the brackets we recognize the determinant $\mathcal{J}^{i_1 \cdots i_k}_{i'_1 \cdots i'_k}$, such that

$$\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = \sum_{1 \leq i'_1 < \cdots < i'_k \leq n} \mathcal{J}^{i_1 \cdots i_k}_{i'_1 \cdots i'_k} \, \mathrm{d}x^{i'_1} \wedge \cdots \wedge \mathrm{d}x^{i'_k}.$$

With all of this follows that the representation in 3.1 can be written as

$$\theta(X) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \theta_{i_1 \cdots i_k}(x^l) \sum_{1 \leq i'_1 < \cdots < i'_k \leq n} \mathcal{J}^{i_1 \cdots i_k}_{i'_1 \cdots i'_k}(x^{l'}) \, \mathrm{d}x^{i'_1} \wedge \cdots \wedge \mathrm{d}x^{i'_k} =$$

$$\sum_{1 \leq i'_1 < \cdots < i'_k \leq n} \left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathcal{J}^{i_1 \cdots i_k}_{i'_1 \cdots i'_k}(x^{l'}) \, \theta_{i_1 \cdots i_k}(X) \right) \mathrm{d}x^{i'_1} \wedge \cdots \wedge \mathrm{d}x^{i'_k}.$$

Compare this with 3.2 and immediately follows the relation 3.3. □

All the given definitions of the tensor fields, in this section, are such that the tensor fields are defined as maps on $\mathbb{R}^n$. Often are tensor fields not defined on the whole $\mathbb{R}^n$ but just on an open subset of it. The same calculation rules remain valid of course.

Section 3.3 Alternative Definition

Let $\Omega \subset \mathbb{R}^n$. Let $\mathcal{K}(\Omega)$ be the set of all coordinate systems on Ω and

$$\mathcal{F}^r_s = \{ F : U \to T^r_s(\mathbb{R}^n) \mid U = f(\Omega), \, f \in \mathcal{K}(\Omega) \}.$$

We assume that the elements of $\mathcal{F}_s{}^r$ are smooth enough. The components of an element $F \in \mathcal{F}_s{}^r$ we note by $F^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ with $1 \leq i_k \leq n$, $1 \leq j_l \leq n$, $1 \leq k \leq r$ and $1 \leq l \leq s$. For coordinate systems we use both the notation f as $\{x^i\}$.

Definition 3.3.1 A $\binom{r}{s}$-tensor field $\mathcal{T}$ on Ω is a map from $\mathcal{K}(\Omega)$ to $\mathcal{F}_s{}^r$ such that if $\mathcal{T} : \{x^i\} \mapsto F$ and $\mathcal{T} : \{x^{i'}\} \mapsto G$ there is satisfied to

$$G^{i'_1 \cdots i'_r}_{j'_1 \cdots j'_s} = \frac{\partial x^{i'_1}}{\partial x^{i_1}}(x^k(x^{k'})) \cdots \frac{\partial x^{i'_r}}{\partial x^{i_r}}(x^k(x^{k'}))\,\frac{\partial x^{j_1}}{\partial x^{j'_1}}(x^{k'}) \cdots \frac{\partial x^{j_s}}{\partial x^{j'_s}}(x^{k'})\, F^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x^k(x^{k'})).$$

□

This means that if, for some curvilinear coordinate system on Ω, a n^{r+s} number of functions on $f(\Omega)$ are given, that there exists just one $\binom{r}{s}$-tensor field on Ω.
It has to be clear that the components of a tensor field out of definition 3.2.4 are the same as the components of a tensor field out of definition 3.3.1, both with respect to the same curvilinear coordinate system.

The alternative definition is important, because one wants to do algebraic and analytical operations, for instance differentation, without to be linked to a fixed chosen coordinate system. If after these calculations a set of functons is obtained, it is the question if these functions are the components of a tensor field. That is the case if they satisfy the transformation rules. Sometimes they are already satisfied if there is satisfied to these transformation rules inside a fixed chosen class (a preferred class) of curvilinear coordinate systems. An example of such a preferred class is the class of the affine coordinate transformations. This class is described by

$$x^{i'} = b^{i'} + L^{i'}_i\, x^i, \tag{3.5}$$

with $\left[b^{i'}\right] \in \mathbb{R}^n$ and $\left[L^{i'}_i\right] \in \mathbb{R}^n_n$ invertible. Coordinates which according 3.5 are associated with the cartesian coordinates are called affine coordinates. Even more important are certain subgroups of it:

i. $\left[L^{i'}_i\right]$ orthogonal: Euclidean invariance, the "Principle of Objectivity' in the continuum mechanics.
ii. $\left[L^{i'}_i\right]$ Lorentz: Lorentz invariance in the special theory of relativity.
iii. $\left[L^{i'}_i\right]$ symplectic: Linear canonical transformations in the classical mechanics.

If inside the preferred class the transformations are valid, than are the components of the obtained tensor field outside the preferred class. An explicit formula is most of the time not given or difficult to obtain.
All the treatments done in the previous chapter can also be done with the tensor fields. They can be done pointswise for every X on the spaces $T_{X^r_s}(\mathbb{R}^n)$.

Section 3.4 Examples of Tensor Fields

3.4.1 The Kronecker Tensor Field

In section 2.3 we introduced the Kronecker tensor. This is the tensor which adds to every basis the identity matrix with mixed indices. Now we define the Kronecker tensor field as the $\binom{1}{1}$-tensor field that adds to every $X \in \mathbb{R}^n$ the Kronecker tensor in $T_{X\,1}^{\;1}(\mathbb{R}^n)$. Because of the fact that

$$\delta^{i'}_{j'}(x^{k'}) = \frac{\partial x^{i'}}{\partial x^{j'}}(x^{k'}) = \frac{\partial x^{i'}}{\partial x^{i}}(x^k(x^{k'}))\,\frac{\partial x^{i}}{\partial x^{j'}}(x^{k'}) = \frac{\partial x^{i'}}{\partial x^{i}}(x^k(x^{k'}))\,\frac{\partial x^{j}}{\partial x^{j'}}(x^{k'})\,\delta^{i}_{j}(x^k(x^{k'}))$$

there is indeed defined a $\binom{1}{1}$-tensor field.

3.4.2 Fundamental Tensor Fields

Let $(\cdot,\cdot)$ be a symmetric inner product on $\mathbb{R}^n$. Let $\mathbf{v}, \mathbf{w} \in T_X(\mathbb{R}^n)$ and define the inner product $(\cdot,\cdot)_X$ by

$$(\mathbf{v},\mathbf{w})_X = \delta_{ij}\, v^i\, w^j.$$

Here are v^i and w^j the components with respect to the elementary basis of $T_X(\mathbb{R}^n)$. These are found on a natural way with the help of the cartesian coordinates. Let $\mathcal{G}_X$ be the isomorphism of $T_X(\mathbb{R}^n)$ to $T^*_X(\mathbb{R}^n)$, which belongs to the inner product $(\cdot,\cdot)_X$. This isomorphism is introduced in Theorem 2.5.1 and is there defined by

$$\mathcal{G}_X : \mathbf{v} \mapsto \widehat{\mathbf{v}}, \text{ with } <\widehat{\mathbf{v}},\mathbf{y}>_X = (\mathbf{v},\mathbf{y})_X.$$

Definition 3.4.1 The fundamental tensor field g is the $\binom{0}{2}$-tensor field on $\mathbb{R}^n$ defined by

$$g(\mathbf{v},\mathbf{w}) = (\mathbf{v},\mathbf{w})_X.$$

□

Lemma 3.4.1 For every curvilinear coordinate system $\{x^i\}$ on $\mathbb{R}^n$ holds

$$g = g_{ij}\,dx^i \otimes dx^j, \text{ with } g_{ij}(x^k) = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)_X. \tag{3.6}$$

Proof There holds

$$\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)_X = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = (g_{kl}\, \mathrm{d}x^k \otimes \mathrm{d}x^l)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{kl}\, \delta_i^k\, \delta_j^l = g_{ij}.$$

There is used that $\mathrm{d}x^p(\frac{\partial}{\partial x^s}) = \delta_s^p$, see also Definition 2.8.12. □

A curvilinear coordinate system $\{x^i\}$ is called an orthogonal curvilinear coordinate system, if at every point X, $[g_{ij}(x^k)]$ is a diagonal matrix. This diagonal matrix is pointwise to tranfer into a diagonal matrix with only the numbers ± 1 on the diagonal. Generally this is not possible for all points simultaneously, because this would impose too many constraints to the curvilinear coordinates. If the chosen inner product is positive, then there can be entered functions h_i such that $g_{ij} = \delta_{ij}\, h_i^2$. These functions are called scale factors.

Comment(s): 3.4.1 The length of the vectors $\frac{1}{h_i}\frac{\partial}{\partial x^i}$ and $h_i\, \mathrm{d}x^i$ (not summate!) are equal to one, because

$$\left|\frac{1}{h_i}\frac{\partial}{\partial x^i}\right| = \sqrt{\left(\frac{1}{h_i}\frac{\partial}{\partial x^i}, \frac{1}{h_i}\frac{\partial}{\partial x^i}\right)_X} = \sqrt{\frac{1}{h_i^2}\, g_{ii}} = 1 \text{ (not summate!)}$$

and

$$\left|h_i\, \mathrm{d}x^i\right| = \sqrt{(h_i\, \mathrm{d}x^i, h_i\, \mathrm{d}x^i)_X} = \sqrt{h_i^2\, g^{ii}} = 1 \text{ (not summate!)}.$$

The bases $\left\{\frac{1}{h_i}\frac{\partial}{\partial x^i}\right\}$ and $\left\{h_i\, \mathrm{d}x^i\right\}$ are orthonormal bases to the corresponding tangent space and its dual.

3.4.3 Volume Forms and Densities

Let $\{x^i\}$ be the cartesian coordinates on $\mathbb{R}^n$ and look to the differential form (n-form) $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, see Definition 3.2.5. The element $X \in \mathbb{R}^n$ is fixed and $\mathbf{v}_1, \cdots, \mathbf{v}_n \in T_X(\mathbb{R}^n)$. The number

$$(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n)(\mathbf{v}_1, \cdots, \mathbf{v}_n)$$

is the oriented volume of a parallellepipedum spanned by $\mathbf{v}_1, \cdots, \mathbf{v}_n \in T_X(\mathbb{R}^n)$. By the transition to curvilinear coordinates $\{x^{i'}\}$ the n-form transforms as

$$\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \frac{\partial(x^1, \cdots, x^n)}{\partial(x^{1'}, \cdots, x^{n'})}\, \mathrm{d}x^{1'} \wedge \cdots \wedge \mathrm{d}x^{n'}$$

such that in general $(\mathrm{d}x^{1'} \wedge \cdots \wedge \mathrm{d}x^{n'})(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ will give another volume than $(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n)(\mathbf{v}_1, \cdots, \mathbf{v}_n)$. If we restrict ourselves to affine coordinate transformations $x^{i'} = b^{i'} + L_i^{i'} x^i$ with $\det L = 1$, than holds $\dfrac{\partial(x^1, \cdots, x^n)}{\partial(x^{1'}, \cdots, x^{n'})} = 1$. In such a case, the volume is called 'invariant' under the given coordinate transformation.
A density is a antisymmetric Tensor Field of the form $\phi'(x^{k'})\mathrm{d}x^{1'} \wedge \cdots \wedge \mathrm{d}x^{n'}$ with a function ϕ', which satisfies

$$\phi'(x^{k'})\mathrm{d}x^{1'} \wedge \cdots \wedge \mathrm{d}x^{n'} = \phi''(x^{k''})\mathrm{d}x^{1''} \wedge \cdots \wedge \mathrm{d}x^{n''},$$

so

$$\phi'(x^{k'}) = \phi''(x^{k''}(x^{k'}))\, \frac{\partial(x^{1''}, \cdots, x^{n''})}{\partial(x^{1'}, \cdots, x^{n'})}.$$

Section 3.5 Examples of Curvilinear Coordinates

3.5.1 Polar coordinates on $\mathbb{R}^2$

Notate the cartesian coordinates on $\mathbb{R}^2$ by x and y, and the polar coordinates by r and ϕ. The cartesian coordinates depend on the polar coordinates by $x = r\cos\phi$ and $y = r\sin\phi$. There holds

$$\begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2 + y^2}} & -y \\ \dfrac{y}{\sqrt{x^2 + y^2}} & x \end{pmatrix},$$

and out of this result follows easily that

$$\begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\dfrac{\sin\phi}{r} & \dfrac{\cos\phi}{r} \end{pmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2 + y^2}} & \dfrac{y}{\sqrt{x^2 + y^2}} \\ -\dfrac{y}{x^2 + y^2} & \dfrac{x}{x^2 + y^2} \end{pmatrix}.$$

With the use of these transition matrices we find the following relations between the bases and dual bases

$$\begin{cases} \dfrac{\partial}{\partial r} = \dfrac{x}{\sqrt{x^2+y^2}}\dfrac{\partial}{\partial x} + \dfrac{y}{\sqrt{x^2+y^2}}\dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial \phi} = -y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y} \end{cases} \qquad \begin{cases} \dfrac{\partial}{\partial x} = \cos\phi\dfrac{\partial}{\partial r} - \dfrac{\sin\phi}{r}\dfrac{\partial}{\partial \phi} \\ \dfrac{\partial}{\partial y} = \sin\phi\dfrac{\partial}{\partial r} + \dfrac{\cos\phi}{r}\dfrac{\partial}{\partial \phi} \end{cases}$$

$$\begin{cases} \mathrm{d}r = \dfrac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}x + \dfrac{y}{\sqrt{x^2+y^2}}\,\mathrm{d}y \\ \mathrm{d}\phi = -\dfrac{y}{x^2+y^2}\,\mathrm{d}x + \dfrac{x}{x^2+y^2}\,\mathrm{d}y \end{cases} \qquad \begin{cases} \mathrm{d}x = \cos\phi\,\mathrm{d}r - r\sin\phi\,\mathrm{d}\phi \\ \mathrm{d}y = \sin\phi\,\mathrm{d}r + r\cos\phi\,\mathrm{d}\phi \end{cases}$$

With the help of these relations are tensor fields, given in cartesian coordinates, to rewrite in other coordinates, for instance polar coordinates. In polar coordinates is the vectorfield

$$\frac{x}{x^2+y^2}\frac{\partial}{\partial x} + \frac{y}{x^2+y^2}\frac{\partial}{\partial y}$$

given by $\dfrac{1}{r}\dfrac{\partial}{\partial r}$, the 2-form $(x^2+y^2)\,\mathrm{d}x \wedge \mathrm{d}y$ by $r^3\,\mathrm{d}r \wedge \mathrm{d}\phi$ and the volume form $\mathrm{d}x \wedge \mathrm{d}y$ by $r\,\mathrm{d}r \wedge \phi$. The fundamental tensor field which belongs to the natural inner product on $\mathbb{R}^2$ can be described in polar coordinates by

$$\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y = \mathrm{d}r \otimes \mathrm{d}r + r^2\,\mathrm{d}\phi \otimes \mathrm{d}\phi. \tag{3.7}$$

3.5.2 Cylindrical coordinates on $\mathbb{R}^3$

Notate the cartesian coordinates on $\mathbb{R}^3$ by x, y and z, and the cylindrical coordinates by r, ϕ and z. The cartesian coordinates depend on the cylindrical coordinates by $x = r\cos\phi$, $y = r\sin\phi$ and $z = z$. The relations between the bases and the dual bases are the same as to the polar coordinates, supplemented with $\mathrm{d}z = \mathrm{d}z$ and $\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial z}$.
The state of stress of a tube under an internal pressure p, such that the axial displacements are prevented, is given by the contravariant 2-tensor field

$$T = \frac{a^2 p}{b^2 - a^2}\left(\left(1 - \frac{b^2}{r^2}\right)\frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} + \left(1 + \frac{b^2}{r^2}\right)\frac{1}{r^2}\frac{\partial}{\partial \phi} \otimes \frac{\partial}{\partial \phi} + 2\nu\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z}\right),$$

where a and b, with $a < b$, are the radii of the inside and outside wall of the tube. Further is ν some material constant.

3.5.3 Spherical coordinates on $\mathbb{R}^3$

Notate the cartesian coordinates on $\mathbb{R}^3$ by x, y and z, and the spherical coordinates by ρ, θ, ϕ. The cartesian coordinates depend on the spherical coordinates by $x = \rho\cos\phi\sin\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\theta$. There holds

$$\begin{pmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\theta & -\rho\sin\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{xz}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{yz}{\sqrt{x^2+y^2}} & x \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & -\sqrt{x^2+y^2} & 0 \end{pmatrix}.$$

After some calculations follows that

$$\begin{pmatrix} \frac{\partial\rho}{\partial x} & \frac{\partial\rho}{\partial y} & \frac{\partial\rho}{\partial z} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} & \frac{\partial\theta}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta \\ \frac{\cos\phi\cos\theta}{\rho} & \frac{\sin\phi\cos\theta}{\rho} & -\frac{\sin\theta}{\rho} \\ -\frac{\sin\phi}{\rho\sin\theta} & \frac{\cos\phi}{\rho\sin\theta} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} & \frac{yz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} & -\frac{\sqrt{x^2+y^2}}{(x^2+y^2+z^2)} \\ -\frac{y}{(x^2+y^2)} & \frac{x}{(x^2+y^2)} & 0 \end{pmatrix}.$$

With the help of these two transition matrices the relations between bases and dual bases can be shown. Tensor Fields expressed in cartesian coordinates can be rewritten in spherical coordinates. So is the volume $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ rewritten in spherical coordinates equal to $\rho^2 \sin\theta \,\mathrm{d}\rho \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$. The electrical field due to a point charge in the originis given by cartesian coordinates by

$$(x^2+y^2+z^2)^{-\frac{3}{2}} \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right)$$

and in spherical coordinates by the simple formula $\frac{1}{\rho^2}\frac{\partial}{\partial\rho}$. Further transforms the fundamental tensor field, corresponding to the natural inner product on $\mathbb{R}^3$, as follows

$$\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z = \mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2\,\mathrm{d}\theta \otimes \mathrm{d}\theta + \rho^2 \sin^2\theta\,\mathrm{d}\phi \otimes \mathrm{d}\phi.$$

The state of stress of a hollw ball under an internal pressure p is given by the contravariant 2-tensor field

$$T = \frac{a^3 p}{b^3 - a^3}\left(\left(1 - \frac{b^3}{\rho^3}\right)\frac{\partial}{\partial\rho} \otimes \frac{\partial}{\partial\rho} + \left(1 + \frac{b^3}{2\,\rho^3}\right)\frac{1}{\rho^2}\frac{\partial}{\partial\theta} \otimes \frac{\partial}{\partial\theta} + \left(1 + \frac{b^3}{2\,\rho^3}\right)\frac{1}{\rho^2\sin^2\theta}\frac{\partial}{\partial\phi} \otimes \frac{\partial}{\partial\phi}\right),$$

where a and b, with $a < b$, are the radii of the inside and outside wall of the ball.

Section 3.6 Differential Operations on Tensor Fields

3.6.1 The gradient

Let f be a scalar field on $\mathbb{R}^n$ and let $\{x^i\}$ be a curvilinear coordinate system on $\mathbb{R}^n$. Let $\{x^{i'}\}$ be another curvilinear coordinate system on $\mathbb{R}^n$. Since

$$\frac{\partial f}{\partial x^{i'}} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial f}{\partial x^i}$$

are the functions $\partial_i f = \frac{\partial f}{\partial x^i}$ the components of a covariant tensor field.

Definition 3.6.1 The covariant tensor field $\mathrm{d}f = \partial_i f\,\mathrm{d}x^i$ is called the <u>gradient field</u> of the scalar field f. □

Let a be a vector field and let a^i be the components of this vector field with respect to the curvilinear coordinates x^i. The functions $a^i\,\partial_j f$ form the components of a $\binom{1}{1}$–tensor field.

Definition 3.6.2 The contraction $a^i\,\partial_i f$ is called the <u>directional derivative</u> of f in the direction a, notation

$$\mathcal{L}_a f = < \mathrm{d}f, \mathbf{a} > = a^i\,\partial_i f.$$

□

If there is defined an inner product on $\mathbb{R}^n$, than there can be formed out of the gradient field, the contravariant vectorfield

$$\mathcal{G}^{-1}\,\mathrm{d}f \;=\; g^{ki}\,\partial_i f\,\frac{\partial}{\partial x^k}.$$

Confusingly enough $\mathcal{G}^{-1}\,\mathrm{d}f$ is often called the 'gradient of f'. If $\{x^i\}$ is an orthogonal curvilinear coordinate system than we can write

$$\mathcal{G}^{-1}\,\mathrm{d}f \;=\; \frac{1}{h_1}\frac{\partial f}{\partial x^1}\frac{1}{h_1}\frac{\partial}{\partial x^1} + \cdots + \frac{1}{h_n}\frac{\partial f}{\partial x^n}\frac{1}{h_n}\frac{\partial}{\partial x^n}.$$

3.6.2 The Lie derivative

Let $\mathbf{v}$ and $\mathbf{w}$ be contravariant vector fields on $\mathbb{R}^n$.

Definition 3.6.3 With respect to the curvilinear coordinates x^i we define

$$(\mathcal{L}_{\mathbf{v}}\,\mathbf{w})^j \;=\; w^i\,\partial_i v^j \;-\; v^i\,\partial_i w^j.$$

□

Let $\{x^{i'}\}$ be some other coordinate system, than holds

$$\begin{aligned}
(\mathcal{L}_{\mathbf{v}}\,\mathbf{w})^{j'} &= w^{i'}\,\partial_{i'} v^{j'} \;-\; v^{i'}\,\partial_{i'} w^{j'} \\
&= A^{i'}_i w^i\,A^k_{i'}\,\partial_k\left(A^{j'}_j\,v^j\right) - A^{i'}_i v^i\,A^k_{i'}\,\partial_k\left(A^{j'}_j\,w^j\right) \\
&= w^k\,\partial_k\left(A^{j'}_j\,v^j\right) - v^k\,\partial_k\left(A^{j'}_j\,w^j\right) \\
&= w^k\left(v^j\,\partial_k A^{j'}_j \;+\; A^{j'}_j\,\partial_k v^j\right) - v^k\left(w^j\,\partial_k A^{j'}_j \;+\; A^{j'}_j\,\partial_k w^j\right) \\
&= A^{j'}_j\,\left(w^k\partial_k v^j \;-\; v^k\,\partial_k w^j\right) + \left(w^k\,v^j \;-\; v^k\,w^j\right)\partial_k A^{j'}_j \\
&= A^{j'}_j\,(\mathcal{L}_{\mathbf{v}}\,\mathbf{w})^j \;+\; w^j\,v^k\left(\partial_j A^{j'}_k \;-\; \partial_k A^{j'}_j\right) \\
&= A^{j'}_j\,(\mathcal{L}_{\mathbf{v}}\,\mathbf{w})^j\,.
\end{aligned}$$

It seems that the functions $(\mathcal{L}_{\mathbf{v}}\,\mathbf{w})^j$ are the components of a contravariant vector field. The vector field $\mathcal{L}_{\mathbf{v}}\,\mathbf{w}$ is called the Lie product of $\mathbf{v}$ and $\mathbf{w}$. With this product the space of vector fields forms a Lie algebra. For a nice geometrical interpretation of the Lie product we refer to (Abraham et al., 2001) ,Manifolds, $\cdots$ or (Misner et al., 1973) ,Gravitation.

3.6.3 Christoffel symbols on $\mathbb{R}^n$

Let $\{x^i\}$ be a curvilinear coordinate system on $\mathbb{R}^n$.

Definition 3.6.4 The n^3 function $\left\{ {i \atop j \quad k} \right\}$ defined by

$$\partial_j \partial_k X = \left\{ {i \atop j \quad k} \right\} \partial_i X,$$

are called Christoffel symbols. □

Notice(s): 3.6.1 The following two equalities are easy to verify

- $\left\{ {i \atop j \quad k} \right\} = \left\{ {i \atop k \quad j} \right\}$ and
- $\left\{ {i \atop j \quad k} \right\} = < dx^i, \partial_j \partial_k X >.$

Let $\{x^{i'}\}$ be another curvilinear coordinate system on $\mathbb{R}^n$, than holds

$$\begin{aligned}
\left\{ {i' \atop j' \quad k'} \right\} &= < dx^{i'}, \partial_{j'} \partial_{k'} X > \\
&= < A^{i'}_i \, dx^i, A^j_{j'} \partial_j \left(A^k_{k'} \partial_k X \right) > \\
&= A^{i'}_i < dx^i, A^j_{j'} A^k_{k'} \partial_j \partial_k X + A^j_{j'} \left(\partial_j A^k_{k'} \right) \partial_k X > \\
&= A^{i'}_i A^j_{j'} A^k_{k'} < dx^i, \partial_j \partial_k X > + A^{i'}_i A^j_{j'} \left(\partial_j A^k_{k'} \right) < dx^i, \partial_k X > \\
&= A^{i'}_i A^j_{j'} A^k_{k'} \left\{ {i \atop j \quad k} \right\} + A^{i'}_i A^j_{j'} \left(\partial_j A^k_{k'} \right) \delta^i_k ,
\end{aligned}$$

what means that

$$\left\{ {i' \atop j' \quad k'} \right\} = A^{i'}_i A^j_{j'} A^k_{k'} \left\{ {i \atop j \quad k} \right\} + A^{i'}_i \partial_{j'} A^i_{k'} . \tag{3.8}$$

The term $A^{i'}_i \partial_{j'} A^i_{k'}$ is in general not equal to zero, so the Christoffel symbols are not the components of a $\binom{1}{2}$-tensor field.

In the definition of the Christoffel symbols there is on no way used an inner product on $\mathbb{R}^n$. If there is defined a symmetric inner product on $\mathbb{R}^n$, than the Christoffel symbols are easy to calculate with the help of the fundamental tensor field. The Christoffel symbols are than to express in the components of the fundamental vector field g_{ij} en its inverse g^{kl}. There holds, see also Lemma 3.6,

$$\begin{aligned}\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} &= \left(\partial_i \partial_j X, \partial_k X\right) + \left(\partial_j X, \partial_i \partial_k X\right) + \\ &\left(\partial_j \partial_k X, \partial_i X\right) + \left(\partial_k X, \partial_j \partial_i X\right) + \\ &- \left(\partial_k \partial_i X, \partial_j X\right) - \left(\partial_i X, \partial_k \partial_j X\right) \\ &= 2 \left(\partial_k X, \partial_i \partial_j X\right).\end{aligned}$$

The inner product can be written as the action of a covector on a vector. So

$$\left(\partial_k X, \partial_i \partial_j X\right) = g_{kl} < \mathrm{d}x^l, \partial_i \partial_j X >= g_{kl} \left\{ {l \atop i \quad j} \right\},$$

out of this follows the identity

$$\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} = 2\, g_{kl} \left\{ {l \atop i \quad j} \right\}.$$

Multiply the obtained identity by $\frac{1}{2}\, g^{mk}$ and then it turns out that

$$\left\{ {m \atop i \quad j} \right\} = \frac{1}{2}\, g^{mk} \left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\right). \tag{3.9}$$

For affine coordinates, which depend on the cartesian coordinates as given in formula 3.5, the Christoffel symbols are all equal to zero. Choose the normal inner product on $\mathbb{R}^n$, then is the Gram matrix $G = I$, see Definition 2.5.2. With Comment 2.5.1 follows that $G_{,,} = L^T L$. Since L is a constant matrix, all the components of the fundamental tensor field are constant. These components correspond to the normal inner product, with respect to the arbitrary affine coordinates. Out of the Identity 3.9 follows directly that all the Christoffel symbols are equal to zero.
With the help of Formula 3.7 follows that for polar coordinates holds that

$$G = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

So we find that

$$\begin{aligned}\left\{ {2 \atop 1 \quad 2} \right\} = \left\{ {2 \atop 2 \quad 1} \right\} &= \frac{1}{2} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \text{ and} \\ \left\{ {1 \atop 2 \quad 2} \right\} &= -r.\end{aligned}$$

All the other Christoffel symbols, which belong to the polar coordinates, are equal to zero.

3.6.4 The covariant derivative on $\mathbb{R}^n$

Let $\mathbf{a}$ be a vector field on $\mathbb{R}^n$ and let $\{x^i\}$ be a curvilinear coordinate system on $\mathbb{R}^n$. Write $\mathbf{a} = a^i \dfrac{\partial}{\partial x^i}$. Let there also be a second curvilinear coordinate system $\{x^{i'}\}$ on $\mathbb{R}^n$, then holds

$$\partial_{j'} a^{i'} = A^j_{j'} \partial_j \left(A^{i'}_i a^i \right) = A^j_{j'} A^{i'}_i \partial_j a^i + a^i \partial_{j'} A^{i'}_i . \tag{3.10}$$

The second term in Formula 3.10 is in general not equal to zero, so the functions $\partial_j a^i$ are not the components of a $\binom{1}{1}$-tensor field.

Definition 3.6.5 We define the n^2 function $\nabla_j a^i$ by

$$\nabla_j a^i = \partial_j a^i + \begin{Bmatrix} & i & \\ j & & k \end{Bmatrix} a^k . \tag{3.11}$$

□

Lemma 3.6.1 The functions $\nabla_j a^i$ form the components of a $\binom{1}{1}$-tensor field.

Proof Because of the transformation rule of the Christoffel symbols, see Formula 3.8, and of Formula 3.10 holds

$$\begin{aligned}
\nabla_{j'} a^{i'} &= \partial_{j'} a^{i'} + \begin{Bmatrix} & i' & \\ j' & & k' \end{Bmatrix} a^{k'} \\
&= A^{i'}_i A^j_{j'} \partial_j a^i + \left(\partial_{j'} A^{i'}_i \right) a^i + \left[A^{i'}_i A^j_{j'} A^k_{k'} \begin{Bmatrix} & i & \\ j & & k \end{Bmatrix} + A^{i'}_i \left(\partial_{j'} A^i_{k'} \right) \right] A^{k'}_l a^l . \\
&= A^{i'}_i A^j_{j'} \left[\partial_j a^i + A^k_{k'} \begin{Bmatrix} & i & \\ j & & k \end{Bmatrix} A^{k'}_l a^l \right] + \left(\partial_{j'} A^{i'}_i \right) a^i + A^{i'}_i \left(\partial_{j'} A^i_{k'} \right) A^{k'}_l a^l \\
&= A^{i'}_i A^j_{j'} \left[\partial_j a^i + \begin{Bmatrix} & i & \\ j & & k \end{Bmatrix} a^k \right] + \left(\partial_{j'} A^{i'}_i \right) a^i + A^{i'}_i A^{k'}_l \left(\partial_{j'} A^i_{k'} \right) a^l
\end{aligned}$$

Think to the simple formula

$$0 = \partial_{j'} \delta^k_l = \partial_{j'} \left(A^i_{k'} A^{k'}_l \right) = A^i_{k'} \left(\partial_{j'} A^{k'}_l \right) + A^{k'}_l \left(\partial_{j'} A^i_{k'} \right),$$

such that

$$\begin{aligned}\nabla_{j'}\, a^{i'} &= A^{i'}_{i}\, A^{j}_{j'}\, \nabla_j\, a^i + \left(\partial_{j'} A^{i'}_{i}\right) a^i - A^{i'}_{i}\, A^{i}_{k'} \left(\partial_{j'} A^{k'}_{l}\right) a^l \\ &= A^{i'}_{i}\, A^{j}_{j'}\, \nabla_j\, a^i + \left(\partial_{j'} A^{i'}_{i}\right) a^i - \delta^{i'}_{k'} \left(\partial_{j'} A^{k'}_{l}\right) a^l \\ &= A^{i'}_{i}\, A^{j}_{j'}\, \nabla_j\, a^i.\end{aligned}$$

□

Definition 3.6.6 The covariant derivative of a vector field $\mathbf{a}$, notation $\nabla\mathbf{a}$, is given by the $\binom{1}{1}$-tensor field

$$\nabla\mathbf{a} = \nabla_j a^i\, dx^j \otimes \frac{\partial}{\partial x^i},$$

where the components $\nabla_j a^i$ are given by Formula 3.11. □

Let α be a covector field on $\mathbb{R}^n$. It is easy to see that the functions $\partial_j \alpha_i$ are not the components of a $\binom{0}{2}$-tensor field. For covector fields we introduce therefore also a covariant derivative.

Lemma 3.6.2 The n^2 functions $\nabla_j \alpha_i$ defined by

$$\nabla_j \alpha_i = \partial_j \alpha_i - \left\{ {k \atop j \quad i} \right\} \alpha_k. \tag{3.12}$$

form the components of $\binom{0}{2}$-tensor field. □

Definition 3.6.7 The covariant derivative of a covector field α, notation $\nabla\alpha$, is given by the $\binom{0}{2}$-tensor field

$$\nabla\alpha = \nabla_j \alpha_i\, dx^j \otimes dx^i,$$

where the components $\nabla_j \alpha_i$ are given by Formula 3.12. □

With the help of the covariant derivative of a vector field, there can be given a definition of the divergence of a vector field.

Definition 3.6.8 The divergence of a vector field a is given by the scalar field $\nabla_i a^i$. The functions a^i are the components of a with respect to some arbitrary curvilinear coordinate system. □

Notice(s): 3.6.2 Because of the fact that the calculation of a covariant derivative is a tensorial operation, it does not matter with respect of what coordinate system the functions a^i are calculated and subsequent to calculate the divergence of $\mathbf{a}$. The fact is that $\nabla_{i'} a^{i'} = \nabla_i a^i$.

With the help of covariant derivative, the gradient field and the fundamental tensor field there can be given a definition of the Laplace operator

Definition 3.6.9 Let ϕ be a scalar field, the Laplace operator, notation $\triangle$, is defined by

$$\triangle\phi = \nabla \mathcal{G}^{-1} \, \mathrm{d}\phi = \nabla_i g^{ij} \, \partial_j \phi.$$

□

Notice(s): 3.6.3 Again the observation that because of the tensorial actions of the various operations, it does not matter what coordinate system is chosen for the calculations.

Later on we come back to the classical vector operations grad, div, rot and $\triangle$. They are looked from some other point of view.

3.6.5 The exterior derivative

A differential form of order k is a tensor field that adds to every point $X \in \mathbb{R}^n$ a anti-symmetric covariant k-tensor in $\bigwedge^k_X(\mathbb{R}^n)$. A differential form of order k is also called a k-form or a antisymmetric k-tensor field, see Definition 3.2.5. There are $(n+1)$ types of non-trival k-forms. These are the 0-forms (the scalarfields), 1-forms (the covector-fields), $\ldots$, n-forms. In Section 2.10 is already commented that antisymmetric k-tensors, with $k > n$, are not interesting types, because they add 0 to every point.

To an arbitrary curvilinear coordinate system $\{x^i\}$ and a k-form ϑ, belong $\binom{n}{k}$ functions $\vartheta_{i_1 \cdots i_k}$, $1 \leq i_1 < \cdots < i_k \leq n$, such that for every $X \in \mathbb{R}^n$ holds

$$\vartheta(X) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \vartheta_{i_1 \cdots i_k}(X) \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}. \tag{3.13}$$

Hereby presents $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ a basisvector of $\bigwedge^k_X(\mathbb{R}^n)$. In Lemma 3.2.1 is described how the functions $\vartheta_{i_1 \cdots i_k}$ transform if there is made a transition to other curvilinear

coordinates.
In this section we define a differentiation operator d, such that k-forms become $(k+1)$-forms, while n-forms become zero.

Lemma 3.6.3 Let $f_1, \cdots, f_r$ be functions of $(r+1)$-variables, notation

$$f_i = f_i(x^1, \cdots, x^{r+1}), \text{ for } i = 1, \cdots, r,$$

then holds

$$\sum_{l=1}^{r+1} (-1)^l \frac{\partial}{\partial x^l} \left(\frac{\partial(f_1, \cdots, f_r)}{\partial(x^1, \cdots, x^{l-1}, x^{l+1}, \cdots, x^{r+1})} \right) = 0. \tag{3.14}$$

□

Proof We give a sketch of the proof. Call $F = (f_1, \cdots, f_r)^T$. The l-th sommand of the summation in the left part of Formula 3.14 is than, on a factor -1, to write as

$$\frac{\partial}{\partial x^l} \det \left(\frac{\partial F}{\partial x^1}, \cdots, \frac{\partial F}{\partial x^{l-1}}, \frac{\partial F}{\partial x^{l+1}}, \cdots, \frac{\partial F}{\partial x^{r+1}} \right) =$$

$$\left| \frac{\partial^2 F}{\partial x^l \partial x^1}, \cdots, \frac{\partial F}{\partial x^{l-1}}, \frac{\partial F}{\partial x^{l+1}}, \cdots, \frac{\partial F}{\partial x^{r+1}} \right| + \cdots$$

$$+ \left| \frac{\partial F}{\partial x^1}, \cdots, \frac{\partial^2 F}{\partial x^{l-1} \partial x^l}, \frac{\partial F}{\partial x^{l+1}}, \cdots, \frac{\partial F}{\partial x^{r+1}} \right| + \left| \frac{\partial F}{\partial x^1}, \cdots, \frac{\partial F}{\partial x^{l-1}}, \frac{\partial^2 F}{\partial x^l \partial x^{l+1}}, \cdots, \frac{\partial F}{\partial x^{r+1}} \right| + \cdots$$

$$+ \left| \frac{\partial F}{\partial x^1}, \cdots, \frac{\partial F}{\partial x^{l-1}}, \frac{\partial F}{\partial x^{l+1}}, \cdots, \frac{\partial^2 F}{\partial x^l \partial x^{r+1}} \right|.$$

In this way Formula 3.14 is to write as a summation of $r\,(r+1)$ terms, ($r\,(r+1)$ is always a even number!), in the form of pairs

$$\pm \det \left(\frac{\partial F}{\partial x^1}, \cdots, \frac{\partial^2 F}{\partial x^k\, \partial x^l}, \cdots, \frac{\partial F}{\partial x^{r+1}} \right),$$

which cancel each other.

Definition 3.6.10 Let $\{x^i\}$ be a curvilinear coordinate system on $\mathbb{R}^n$ and let ϑ be a k-form. Write ϑ as in Formula 3.13. The exterior derivative of ϑ, notation $d\vartheta$, is defined by

$$d\vartheta = \sum_{r=1}^{n}\left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\partial \vartheta_{i_1 \cdots i_k}}{\partial x^r} dx^r \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right).$$

Note that a summand, where r is equal to one of the i_j's, is equal to zero. The sum formed by the terms, where r is not equal to one of the i_j's, is obviously to write as

$$d\vartheta = \sum_{1 \leq j_1 < \cdots < j_{k+1} \leq n} (d\vartheta)_{j_1 \cdots j_{k+1}} dx^{j_1} \wedge \cdots \wedge dx^{j_{k+1}}$$

□

Note that the exterior derivative of a n-forms is indeed 0. At this moment, there is still the question if $d'\vartheta$, this is the exterior derivative of ϑ with respect to the coordinates $\{x^{i'}\}$, is the same as $d\vartheta$.

Example(s): 3.6.1

- Consider $\mathbb{R}^2$ with the Cartesian coordinates x and y. Let ϕ be a scalar field than is

  $$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

  a 1-form. Let α be a covector field en write $\alpha = \alpha_1 dx + \alpha_2 dy$ than is

  $$d\alpha = \left(\frac{\partial \alpha_2}{\partial x} - \frac{\partial \alpha_1}{\partial y}\right) dx \wedge dy$$

  a 2-form. Let γ be 2-form en write $\gamma = \gamma_{12} dx \wedge dy$ than is $d\gamma = 0$. Note that in all the cases applying d twice always gives zero.

Example(s): 3.6.2

- Consider $\mathbb{R}^3$ with the Cartesian coordinates x, y and z. Let ϕ be a scalar field than is

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

a 1-form. Let α be a covector field en write $\alpha = \alpha_1 \, dx + \alpha_2 \, dy + \alpha_3 \, dz$ than is

$$d\alpha = \left(\frac{\partial \alpha_1}{\partial x} dx + \frac{\partial \alpha_1}{\partial y} dy + \frac{\partial \alpha_1}{\partial z} dz\right) \wedge dx +$$
$$\left(\frac{\partial \alpha_2}{\partial x} dx + \frac{\partial \alpha_2}{\partial y} dy + \frac{\partial \alpha_2}{\partial z} dz\right) \wedge dy + \left(\frac{\partial \alpha_3}{\partial x} dx + \frac{\partial \alpha_3}{\partial y} dy + \frac{\partial \alpha_3}{\partial z} dz\right) \wedge dz =$$
$$\left(\frac{\partial \alpha_2}{\partial x} - \frac{\partial \alpha_1}{\partial y}\right) dx \wedge dy + \left(\frac{\partial \alpha_3}{\partial x} - \frac{\partial \alpha_1}{\partial z}\right) dx \wedge dz + \left(\frac{\partial \alpha_3}{\partial y} - \frac{\partial \alpha_2}{\partial z}\right) dy \wedge dz$$

a 2-form. Let ω be a covector field en write $\omega = \omega_{12} \, dx \wedge dy + \omega_{13} \, dx \wedge dz + \omega_{23} \, dy \wedge dz$ than is

$$\begin{aligned} d\omega &= \frac{\partial \omega_{12}}{\partial z} dz \wedge dx \wedge dy + \frac{\partial \omega_{13}}{\partial y} dy \wedge dx \wedge dz + \frac{\partial \omega_{23}}{\partial x} dx \wedge dy \wedge dz \\ &= \left(\frac{\partial \omega_{23}}{\partial x} - \frac{\partial \omega_{13}}{\partial y} + \frac{\partial \omega_{12}}{\partial z}\right) dx \wedge dy \wedge dz \end{aligned}$$

a 3-form. Let γ be 3-form en write $\gamma = \gamma_{123} \, dx \wedge dy \wedge dz$ than is $d\gamma = 0$. Note that in all the cases applying d twice always gives zero.

Theorem 3.6.1 The definition of the exterior derivative d is independent of the coordinate system. □

Proof Let $\{x^i\}$ and $\{x^{i'}\}$ be two coordinate systems. We prove the proposition for the differential form

$$\omega = \alpha \, dx^1 \wedge \cdots \wedge dx^k,$$

where α is an arbitrary function of the variables x^i. The approach to prove the proposition for an arbitrary differential form of the form $\alpha \, dx^{i_1} \wedge \cdots \wedge di_k$ is analog. The proposition follows by taking linear combinations.
The exterior derivative of ω with respect to the variables x^i is given by

$$d\omega = \sum_{r=1}^{n} \frac{\partial \alpha}{\partial x^r} dx^r \wedge dx^1 \wedge \cdots \wedge dx^k.$$

On the basis of Lemma 3.2.1, ω can be written, with respect to the coordinates $x^{i'}$, as

$$\omega = \sum_{1 \leq i'_1 < \cdots < i'_k \leq n} \alpha \frac{\partial\left(x^1, \cdots, x^k\right)}{\partial\left(x^{i'_1}, \cdots, x^{i'_k}\right)} \mathrm{d}x^{i'_1} \wedge \cdots \wedge \mathrm{d}x^{i'_k}.$$

The exterior derivative of ω with respect to the coordinates $x^{i'}$, notated by $\mathrm{d}'\omega$, is given by

$$\mathrm{d}'\omega = \sum_{r'=1}^{n}\left(\sum_{1 \leq i'_1 < \cdots < i'_k \leq n} \frac{\partial \alpha}{\partial x^{r'}} \frac{\partial\left(x^1, \cdots, x^k\right)}{\partial\left(x^{i'_1}, \cdots, x^{i'_k}\right)} \mathrm{d}x^{r'} \wedge \mathrm{d}x^{i'_1} \wedge \cdots \wedge \mathrm{d}x^{i'_k}\right) + \tag{3.15}$$

$$\alpha \sum_{r'=1}^{n}\left(\sum_{1 \leq i'_1 < \cdots < i'_k \leq n} \frac{\partial}{\partial x^{r'}}\left(\frac{\partial\left(x^1, \cdots, x^k\right)}{\partial\left(x^{i'_1}, \cdots, x^{i'_k}\right)}\right) \mathrm{d}x^{r'} \wedge \mathrm{d}x^{i'_1} \wedge \cdots \wedge \mathrm{d}x^{i'_k}\right). \tag{3.16}$$

The first sum, see Formula 3.15, is to write as (with the use of the index notation)

$$\frac{\partial \alpha}{\partial x^{r'}} \mathrm{d}x^{r'} \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k =$$
$$\frac{\partial x^r}{\partial x^{r'}} \frac{\partial x^{r'}}{\partial x^l} \frac{\partial \alpha}{\partial x^r} \mathrm{d}x^l \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k =$$
$$\frac{\partial \alpha}{\partial x^r} \mathrm{d}x^r \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k,$$

and that we recognize as $\mathrm{d}\omega$. The second sum, see Formula 3.16, is to write as

$$\alpha \sum_{1 \leq j'_1 < \cdots < j'_{k+1} \leq n}\left(\sum_{\{r', i'_1 < \cdots < i'_k\} = \{j'_1 < \cdots < j'_{k+1}\}} \frac{\partial}{\partial x^{r'}}\left(\frac{\partial\left(x^1, \cdots, x^k\right)}{\partial\left(x^{i'_1}, \cdots, x^{i'_k}\right)}\right) \mathrm{d}x^{r'} \wedge \mathrm{d}x^{i'_1} \wedge \cdots \wedge \mathrm{d}x^{i'_k}\right) \tag{3.17}$$

where the inner sum is a summation over all possible combinations $r', i'_1 < \cdots < i'_{k'}$, a collection of $(k+1)$ natural numbers, which coincides with the collection $j'_1 < \cdots < j'_{k+1}$. The inner sum of Formula 3.17 can then be written as

$$\sum_{l=1}^{k+1} \frac{\partial}{\partial x^{j'_l}}\left(\frac{\partial\left(x^1, \cdots, x^k\right)}{\partial\left(x^{j'_1}, \cdots, x^{j'_{l-1}}, x^{j'_{l+1}}, \cdots, x^{j'_{k+1}}\right)}\right) \mathrm{d}x^{j'_l} \wedge \mathrm{d}x^{j'_1} \wedge \cdots \wedge \mathrm{d}x^{j'_{l-1}} \wedge \mathrm{d}x^{j'_{l+1}} \wedge \cdots \wedge \mathrm{d}x^{j'_{k+1}}.$$

Put the obtained $(k+1)$-form $\mathrm{d}x^{j'_l} \wedge \mathrm{d}x^{j'_1} \wedge \cdots \wedge \mathrm{d}x^{j'_{l-1}} \wedge \mathrm{d}x^{j'_{l+1}} \wedge \cdots \wedge \mathrm{d}x^{j'_{k+1}}$ in the order $\mathrm{d}x^{j'_1} \wedge \cdots \wedge \mathrm{d}x^{j'_{k+1}}$. This costs a factor $(-1)^{(l+1)}$. With the use of Lemma 3.6.3 it follows that the second sum, see Formula 3.16, is equal to zero. □

Theorem 3.6.2 Two times applying d on a k-form gives zero, that means $\mathrm{d} \wedge \mathrm{d} = 0$. □

Proof Because of Theorem 3.6.1, it is enough to prove the proposition just for one coordinate system $\{x^i\}$. The same as in the foregoing theorem it is enough to prove the proposition for the k-form

$$\omega = \alpha \, dx^1 \wedge \cdots \wedge dx^k,$$

with α an arbitrary function of the variables x^i. There is

$$d \wedge d\omega = \sum_{l=1}^{n} \sum_{r=1}^{n} \frac{\partial^2 \alpha}{\partial x^l \, \partial x^r} dx^r \wedge dx^l \wedge dx^1 \wedge \cdots \wedge dx^k.$$

This summation exist out of $n\,(n - 1)$ terms, an even number of terms. These terms become pairwise zero, because

$$\frac{\partial^2 \alpha}{\partial x^l \, \partial x^r} = \frac{\partial^2 \alpha}{\partial x^r \, \partial x^l} \text{ and } dx^r \wedge dx^l = -\,dx^l \wedge dx^r.$$

□

Theorem 3.6.2 is the generalisation to $\mathbb{R}^n$ of the classical results

$$\text{rot}\,\text{grad} = 0 \quad \text{and} \quad \text{div}\,\text{rot} = 0 \text{ in } \mathbb{R}^3.$$

Theorem 3.6.3 Let a be a l-form and b be a m-form then is $da \wedge b$ a $(l + m + 1)$-form and there holds

$$d(a \wedge b) = da \wedge b + (-1)^l \, a \wedge db \, .$$

□

Proof We prove the theorem for the special case that

$$a = \alpha \, dx^1 \wedge \cdots \wedge dx^l \text{ and } b = \beta \, dx^{l+1} \wedge \cdots \wedge dx^m,$$

where x^i are arbitrary (curvilinear) coordinates and α and β are functions of the variables x^i. There holds that

$$a \wedge b = \alpha \, \beta \, dx^1 \wedge \cdots \wedge dx^{l+m},$$

such that

$$d(a \wedge b) = \sum_{p=1}^{n} \left(\beta \frac{\partial \alpha}{\partial x^p} + \alpha \frac{\partial \beta}{\partial x^p} \right) dx^p \wedge dx^1 \wedge \cdots \wedge dx^{l+m}$$

Furthermore holds

$$d\alpha \wedge \beta = \sum_{p=1}^{n} \beta \frac{\partial \alpha}{\partial x^p} dx^p \wedge dx^1 \wedge \cdots \wedge dx^{l+m}$$

and

$$\alpha \wedge \mathrm{d}\beta = \sum_{p=1}^{n} \alpha \frac{\partial \beta}{\partial x^p} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^l \wedge \mathrm{d}x^p \wedge \mathrm{d}x^{l+1} \wedge \cdots \wedge \mathrm{d}x^{l+m}.$$

In this last expression it costs a factor $(-1)^l$ to get $\mathrm{d}x^p$ to the front of that expression. Hereby is the theorem proven for the special case. □

Section 3.7 Combinations of the exterior derivative and the Hodge transformation

If there are chosen a symmetric inner product and a oriented volume on $\mathbb{R}^n$, they can be transferred to every tangent space (see the Subsections 3.4.2 and 3.4.3). In every point X can then the Hodge image $*\vartheta(X)$ be considered. This Hodge image is then an antisymmetric $(n-k)$-tensor (see Section 2.12). In this section we consider combinations of the algebraic operator $*$ and the differential operator d on differential forms.

3.7.1 Combinations of d and $*$ in $\mathbb{R}^2$

Let x and y be Cartesian coordinates on $\mathbb{R}^2$ and take the natural inner product.
If $\alpha = \alpha_1\,\mathrm{d}x + \alpha_2\,\mathrm{d}y$, than holds

$$\begin{aligned}
*\,\alpha &= -\alpha_2\,\mathrm{d}x + \alpha_1\,\mathrm{d}y \\
\mathrm{d}*\alpha &= \left(\frac{\partial\alpha_1}{\partial x} + \frac{\partial\alpha_2}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y, \\
\,\mathrm{d}\alpha &= \frac{\partial\alpha_1}{\partial x} + \frac{\partial\alpha_2}{\partial y}.
\end{aligned}$$

Let f be scalar field, than holds

$$\begin{aligned}
\mathrm{d}f &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y, \\
*\,\mathrm{d}f &= -\frac{\partial f}{\partial y}\mathrm{d}x + \frac{\partial f}{\partial x}\mathrm{d}y, \\
\mathrm{d}*\mathrm{d}f &= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x\wedge\mathrm{d}y, \\
\,\mathrm{d}\mathrm{d}f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.
\end{aligned}$$

This last result is the Laplace operator with respect to the natural inner product and volume form on $\mathbb{R}^2$.

3.7.2 Combinations of d and $*$ in $\mathbb{R}^3$

Consider $\mathbb{R}^3$ with the Cartesian coordinates x, y and z and the usual inner product. If $\alpha = \alpha_1 \,\mathrm{d}x + \alpha_2 \,\mathrm{d}y + \alpha_3 \,\mathrm{d}z$, than holds

$$
\begin{aligned}
\mathrm{d}\alpha &= \left(\frac{\partial \alpha_2}{\partial x} - \frac{\partial \alpha_1}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial \alpha_3}{\partial x} - \frac{\partial \alpha_1}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}z + \left(\frac{\partial \alpha_3}{\partial y} - \frac{\partial \alpha_2}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z,\\
* \,\mathrm{d}\alpha &= \left(\frac{\partial \alpha_3}{\partial y} - \frac{\partial \alpha_2}{\partial z}\right) \mathrm{d}x + \left(\frac{\partial \alpha_1}{\partial z} - \frac{\partial \alpha_3}{\partial x}\right) \mathrm{d}y + \left(\frac{\partial \alpha_2}{\partial x} - \frac{\partial \alpha_1}{\partial y}\right) \mathrm{d}z,\\
* \,\alpha &= \alpha_1 \,\mathrm{d}y \wedge \mathrm{d}z - \alpha_2 \,\mathrm{d}x \wedge \mathrm{d}z + \alpha_3 \,\mathrm{d}x \wedge \mathrm{d}y,\\
\mathrm{d} * \alpha &= \left(\frac{\partial \alpha_1}{\partial x} + \frac{\partial \alpha_2}{\partial y} + \frac{\partial \alpha_3}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,\\
* \,\mathrm{d} * \alpha &= \frac{\partial \alpha_1}{\partial x} + \frac{\partial \alpha_2}{\partial y} + \frac{\partial \alpha_3}{\partial z}.
\end{aligned}
$$

Let f be scalar field, than holds

$$
\begin{aligned}
\mathrm{d}f &= \frac{\partial f}{\partial x} \mathrm{d}x + \frac{\partial f}{\partial y} \mathrm{d}y + \frac{\partial f}{\partial z} \mathrm{d}z,\\
* \,\mathrm{d}f &= \frac{\partial f}{\partial z} \mathrm{d}x \wedge \mathrm{d}y - \frac{\partial f}{\partial y} \mathrm{d}x \wedge \mathrm{d}z + \frac{\partial f}{\partial x} \mathrm{d}y \wedge \mathrm{d}z,\\
\mathrm{d} * \mathrm{d}f &= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,\\
* \,\mathrm{d} * \mathrm{d}f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.
\end{aligned}
$$

Also in $\mathbb{R}^3$, the operator $* \mathrm{d} * \mathrm{d}$ seems to be the Laplace operator for scalar fields.

Notice(s): 3.7.1 All the combinations of d and $*$ are coordinate free and can be written out in any desired coordinate system.

3.7.3 Combinations of d and $*$ in $\mathbb{R}^n$

Let there are chosen a symmetric inner product and a matching oriented volume on $\mathbb{R}^n$ and let ω be a k-form.

Definition 3.7.1 The Laplace operator $\triangle$ for ω is defined by

$$\triangle\omega = (-1)^{nk}\,(*\,\mathrm{d} * \mathrm{d}\omega + (-1)^n\,\mathrm{d} * \mathrm{d} * \omega)\,.$$

□

Notice that k-forms become k-forms. In $\mathbb{R}^3$ holds that

$$\triangle(\alpha_1\,\mathrm{d}x + \alpha_2\,\mathrm{d}y + \alpha_3\,\mathrm{d}z) = (\triangle\alpha_1)\,\mathrm{d}x + (\triangle\alpha_2)\,\mathrm{d}y + (\triangle\alpha_3)\,\mathrm{d}z.$$

Check this. Check furthermore that in $\mathbb{R}^4$ with the Lorentz inner product, for scalar fields ϕ, $\triangle\,\phi$ is the same as $\square\,\phi$, where $\square$ represents the d'Alembertian.

Comment(s): 3.7.1 d'Alemertian is also called the the Laplace operator of the Minkowski space. In standard coordinates t, x, y and z and if the inner product has the signature $(+,-,-,-)$, it has the form

$$\square = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2},$$

see also Example 2.12.3.

Section 3.8 The classical vector operations in $\mathbb{R}^3$

These classical vector operations are grad, div, curl and $\triangle$ have only to do with scalar fields and vector fields. In this section we give coordinate free definitons of these operations. Hereby will, beside the operators d and $*$, also the isomorphism $\mathcal{G}_X : T_X(\mathbb{R}^n) \rightarrow T_X^*(\mathbb{R}^n)$ play a role of importance. This is determined by the chosen inner product, see Subsection 3.4.2. We consider here the usual inner product on $\mathbb{R}^3$and the orthogonal coordinates $\{x^i\}$. Furthermore we use the scale factors h_i. With the help of these scale factors, the components of the fundamental tensor field g_{ij} can be written as $g_{ij} = \delta_{ij}\,h_i^2$. Furthermore are the bases $\left\{\frac{1}{h_i}\frac{\partial}{\partial x^i}\right\}$ and $\left\{h_i\,\mathrm{d}x^i\right\}$ orthonormal in every tangent space $T_X(\mathbb{R}^n)$ and its dual. Further holds that $\mathcal{G}_X\left(\frac{1}{h_i}\frac{\partial}{\partial x^i}\right) = h_i\,\mathrm{d}x^i$, wherein may not be summed over i.

3.8.1 The gradient

Let ϕ be a scalar field.

Definition 3.8.1 The gradient of ϕ is the vector field $\operatorname{grad}\phi$, defined by $\operatorname{grad}\phi = \mathcal{G}^{-1}\,\mathrm{d}\phi$. □

The components of $\operatorname{grad}\phi$, belonging to the coordinates x^i, are given by $(\operatorname{grad}\phi)^i = g^{ij}\dfrac{\partial\phi}{\partial x^j}$. In the classical literature it is customary, when there are used orthogonal curvilinear coordinates, to give the components of vector fields with respect to orthonormal bases. Because $g^{ij} = h_i^{-2}\,\delta_{ij}$, is the gradient of ϕ with respect to the orthonormal base $\left\{\dfrac{1}{h_i}\dfrac{\partial}{\partial x^i}\right\}$ to write as

$$\operatorname{grad}\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial x^1}\frac{1}{h_1}\frac{\partial}{\partial x^1} + \frac{1}{h_2}\frac{\partial\phi}{\partial x^2}\frac{1}{h_2}\frac{\partial}{\partial x^2} + \frac{1}{h_3}\frac{\partial\phi}{\partial x^3}\frac{1}{h_3}\frac{\partial}{\partial x^3}.$$

Apparently are the components, with respect to this base, given by $\dfrac{1}{h_i}\dfrac{\partial\phi}{\partial x^i}$.

3.8.2 The curl

Let α be a vector field.

Definition 3.8.2 The curl of α is the vector field $\operatorname{curl}\alpha$, defined by $\operatorname{curl}\alpha = \mathcal{G}^{-1} * \mathrm{d}\mathcal{G}\alpha$. □

We work out the curl of α for orthogonal coordinates x^i. Write

$$\alpha = \alpha^1\frac{1}{h_1}\frac{\partial}{\partial x^1} + \alpha^2\frac{1}{h_2}\frac{\partial}{\partial x^2} + \alpha^3\frac{1}{h_3}\frac{\partial}{\partial x^3},$$

than holds

$$\mathcal{G}\alpha = \alpha^1 h_1\,\mathrm{d}x^1 + \alpha^2 h_2\,\mathrm{d}x^2 + \alpha^3 h_3\,\mathrm{d}x^3,$$

such that

$$\mathrm{d}\mathcal{G}\alpha = \left(\frac{\partial\,\alpha^2 h_2}{\partial x^1} - \frac{\partial\,\alpha^1 h_1}{\partial x^2}\right)\mathrm{d}x^1\wedge\mathrm{d}x^2 + \left(\frac{\partial\,\alpha^3 h_3}{\partial x^1} - \frac{\partial\,\alpha^1 h_1}{\partial x^3}\right)\mathrm{d}x^1\wedge\mathrm{d}x^3 +$$
$$\left(\frac{\partial\,\alpha^3 h_3}{\partial x^2} - \frac{\partial\,\alpha^2 h_2}{\partial x^3}\right)\mathrm{d}x^2\wedge\mathrm{d}x^3.$$

To calculate the Hodge image of $d\mathcal{G}\,\alpha$, we want that the basis vectors are orthonormal. Therefore we write $dx^1 \wedge dx^2 = \frac{1}{h_1 h_2}\left(h_1 dx^1 \wedge h_2 dx^2\right)$, than follows that $* dx^1 \wedge dx^2 = \frac{h_3}{h_1 h_2} dx^3$. With a similar notation for the $dx^1 \wedge dx^3$ and $dx^2 \wedge dx^3$ it follows that

$$
\begin{aligned}
* \, d\mathcal{G}\,\alpha = & \frac{1}{h_2 h_3}\left(\frac{\partial\,\alpha^3 h_3}{\partial x^2} - \frac{\partial\,\alpha^2 h_2}{\partial x^3}\right) h_1 dx^1 + \frac{1}{h_1 h_3}\left(\frac{\partial\,\alpha^1 h_1}{\partial x^3} - \frac{\partial\,\alpha^3 h_3}{\partial x^1}\right) h_2 dx^2 \\
& \frac{1}{h_1 h_2}\left(\frac{\partial\,\alpha^2 h_2}{\partial x^1} - \frac{\partial\,\alpha^1 h_1}{\partial x^2}\right) h_3 dx^3,
\end{aligned}
$$

such that we finally find that

$$
\begin{aligned}
\operatorname{curl}\alpha = & \frac{1}{h_2 h_3}\left(\frac{\partial\,\alpha^3 h_3}{\partial x^2} - \frac{\partial\,\alpha^2 h_2}{\partial x^3}\right) \frac{1}{h_1}\frac{\partial}{\partial x^1} + \frac{1}{h_1 h_3}\left(\frac{\partial\,\alpha^1 h_1}{\partial x^3} - \frac{\partial\,\alpha^3 h_3}{\partial x^1}\right) \frac{1}{h_2}\frac{\partial}{\partial x^2} + \\
& \frac{1}{h_1 h_2}\left(\frac{\partial\,\alpha^2 h_2}{\partial x^1} - \frac{\partial\,\alpha^1 h_1}{\partial x^2}\right) \frac{1}{h_3}\frac{\partial}{\partial x^3}.
\end{aligned}
$$

3.8.3 The divergence

Let α be a vector field.

Definition 3.8.3 The divergence of α is the scalar field $\operatorname{div}\alpha$, defined by $\operatorname{div}\alpha = * d * \mathcal{G}\,\alpha$. □

We write again

$$\alpha = \alpha^1 \frac{1}{h_1}\frac{\partial}{\partial x^1} + \alpha^2 \frac{1}{h_2}\frac{\partial}{\partial x^2} + \alpha^3 \frac{1}{h_3}\frac{\partial}{\partial x^3},$$

than holds

$$* \mathcal{G}\,\alpha = \alpha^1 h_2 h_3 \, dx^2 \wedge dx^3 - \alpha^2 h_1 h_3 \, dx^1 \wedge dx^3 + \alpha^3 h_1 h_2 \, dx^1 \wedge dx^2,$$

such that

$$d * \mathcal{G}\,\alpha = \left(\frac{\partial\,\alpha^1 h_2 h_3}{\partial x^1} + \frac{\partial\,\alpha^2 h_1 h_3}{\partial x^2} + \frac{\partial\,\alpha^3 h_1 h_2}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3$$

and so we get

$$\operatorname{div}\alpha = \frac{1}{h_1 h_2 h_2}\left(\frac{\partial\,\alpha^1 h_2 h_3}{\partial x^1} + \frac{\partial\,\alpha^2 h_1 h_3}{\partial x^2} + \frac{\partial\,\alpha^3 h_1 h_2}{\partial x^3}\right).$$

This is the well-known formula, which will also be found, if the divergence of α is written out as given in Definition 3.6.8.

3.8.4 The Laplace operator

This differential operator we define here for scalar fields and for vector fields.

Definition 3.8.4 Let ϕ be a scalar field, than the Laplace operator for ϕ, notation $\triangle\,\phi$, is defined by

$$\triangle\,\phi \;=\; \operatorname{div}\operatorname{grad}\phi \;=\; *\mathrm{d} * \mathrm{d}\phi\,.$$

□

Note that out of Definition 3.7.1 follows that $\triangle\,\phi \;=\; (*\mathrm{d} * \mathrm{d} - \mathrm{d} * \mathrm{d}\,*)\,\phi$. The second term gives zero, such the above given definition is consistent with Definition 3.7.1.

Definition 3.8.5 Let α be a vector field, than the Laplace operator for α, notation $\triangle\,\alpha$, is defined by

$$\triangle\,\alpha \;=\; -\mathcal{G}^{-1}\,(*\mathrm{d} * \mathrm{d} - \mathrm{d} * \mathrm{d}\,*)\,\mathcal{G}\,\alpha.$$

□

Note that $\operatorname{grad}\operatorname{div}\alpha \;=\; \mathcal{G}^{-1}\,\mathrm{d} * \mathrm{d} * \mathcal{G}$ and that $\operatorname{curl}\operatorname{curl}\alpha \;=\; \mathcal{G}^{-1} * \mathrm{d} * \mathrm{d}\mathcal{G}\alpha$, hereby follows that the above given definition is consistent with the classical formula

$$\triangle\,\alpha \;=\; \operatorname{grad}\operatorname{div}\alpha \;-\; \operatorname{curl}\operatorname{curl}\alpha\,.$$

All formulas out of the classical vector analysis are in such a way to prove. See (Abraham et al., 2001) ,Manifolds, $\cdots$, page 379, Exercise 6.4B.

Section 3.9 Exercises

1. Determine the Christoffel symbols belonging to the cylindrical and spherical coordinates on $\mathbb{R}^3$.

2. Prove Lemma 3.6.2.

Section 3.10 RRvH: Overall example(s)

3.10.1 Helicoidal coordinates

Define $\underline{x} = (x, y, z) \in \mathbb{R}^3$, with the following coordinates

$$\begin{cases} x = \xi \cos\theta + \eta \sin\theta, \\ y = -\xi \sin\theta + \eta \cos\theta, \\ z = \theta. \end{cases}$$

The determinant of the Jacobian matrix is given by

$$J = \left| \frac{\partial \underline{x}}{\partial \xi} \; \frac{\partial \underline{x}}{\partial \eta} \; \frac{\partial \underline{x}}{\partial \theta} \right| = \begin{vmatrix} \cos\theta & \sin\theta & -\xi \sin\theta + \eta \sin\theta \\ -\sin\theta & \cos\theta & -\xi \cos\theta - \eta \sin\theta \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

The inverse coordinate transformation is given by

$$\begin{cases} \xi = x \cos z - y \sin z, \\ \eta = x \sin z + y \cos z, \\ \theta = z. \end{cases}$$

Chapter 4 Differential Geometry

Section 4.1 Differential geometry of curves in $\mathbb{R}^3$

4.1.1 Space curves

With respect to the standard basis $\{E_i\}$ is every point $X \in \mathbb{R}^3$ to write as $X = x^i E_i$. Let now x^i be real functions of a real parameter t, where t runs through a certain interval I. We suppose that the functions x^i are enough times differentiable, such that in the future no difficulties arise with respect to differentiation. Further we assume that the derivatives $\frac{dx^i}{dt}$ are not simultaneously equal to zero, for any value of $t \in I$.

Definition 4.1.1 A space curve K is the set of points $X = X(t) = x^i(t) E_i$. Hereby runs t through the interval I. De map $t \mapsto X(t)$ is injective and smooth enough. □

We call the representation $x^i(t)$ of the space curve K a parameter representation. The vector $\frac{dx^i}{dt} E_i$ is the tangent vector to the space curve at the point X, which will also be written as $\frac{dX}{dt}$. Another parametrisation of K can be obtained by replacing t by $f(u)$. Hereby is f a monotonic function, such that $f(u)$ runs through the interval I, if the parameter u runs through some interval J. Also the function f is expected to be enough times differentiable and such that the first order derivative is not equal to zero for any value of u. We call the transition to another parametric representation, by the way of $t = f(u)$, a parameter transformation. Note that there are infinitely many parametric representations of one and the same space curve K.

Example(s): 4.1.1 A circular helix can be parametrised by

$$x^1(t) = a \cos t,\ x^2(t) = a \sin t, x^3(t) = h\, t,$$

where a and h are constants and t basically runs through $\mathbb{R}$.

The arclength of a (finite) curve K (a finite curve is also called arc) described by the parametric representation $x^i(t)$, with $t_0 \leq \tau \leq t$, is given by

$$s(t) = \int_{t_0}^{t} \sqrt{\left(\frac{dx^1(\tau)}{d\tau}\right)^2 + \left(\frac{dx^2(\tau)}{d\tau}\right)^2 + \left(\frac{dx^3(\tau)}{d\tau}\right)^2}\, d\tau. \tag{4.1}$$

The introduced function s, we want to use as parameter for space curves. The parameter is then s and is called arclength. The integral given in 4.1 is most often difficult, or not all, to determine. Therefore we use the arclength parametrisation of a space curve only for theoretical purposes.

Henceforth we consider the Euclidean inner product on $\mathbb{R}^3$. Out of the main theorem of the integration follows than that

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \left(\frac{\mathrm{d}X}{\mathrm{d}t}, \frac{\mathrm{d}X}{\mathrm{d}t}\right). \tag{4.2}$$

The derivative of s to t is the length of the tangent vector. If s is chosen as parameter of K, than holds

$$\left(\frac{\mathrm{d}X}{\mathrm{d}s}, \frac{\mathrm{d}X}{\mathrm{d}s}\right) = \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \left(\frac{\mathrm{d}X}{\mathrm{d}t}, \frac{\mathrm{d}X}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = 1, \tag{4.3}$$

where we used Formula 4.2. Property 4.3 makes the use of the arclength as parameter so special. In future we use for the vector $\frac{\mathrm{d}X}{\mathrm{d}s}$ a special notation, namely $\dot{X}$.

Example(s): 4.1.2 Look to the circular helix, as introduced in Example 4.1.1, with the start value $t = 0$. There holds that $s(t) = t\sqrt{a^2 + h^2}$.
Note that indeed $\left(\dot{X}, \dot{X}\right) = 1$.

The tangent vector $\dot{X}$ is the tangent vector to K, with length 1.

Definition 4.1.2 The tangent line to a curve K at the point X is straight line given by the parametric representation

$$Y = X + \lambda\,\dot{X}.$$

□

The parameter λ in this definition is in such a way that $|\lambda|$ gives the distance from the tangent point X along this tangent line.

Definition 4.1.3 The osculation plane to a curve K at the point X is the plane that is given by the parametric representation

$$Y = X + \lambda\,\dot{X} + \mu\,\ddot{X}.$$

□

The, in Definition 4.1.3, given parametric representation of the osculation plane is equivalent to the equation

$$\det\left(Y - X, \dot{X}, \ddot{X}\right) = 0.$$

Here we assumed that $\dot{X}$ and $\ddot{X}$ are linear independent and in particular that $\ddot{X} \neq 0$. Geometrically it means that X is no inflection point. Also in inflection points can be defined an osculation plane. An equation of the osculation plane in a inflection point X is given by

$$\det\left(Y - X, \dot{X}, \frac{d^3 X}{ds^3}\right) = 0.$$

For some arbitrary parametrisation of a space curve K, with parameter t, the parameter representation of the tangent line and the osculation plane in X, with $\ddot{X} \neq 0$, are given by

$$Y = X + \lambda \frac{dX}{dt}$$

and

$$Y = X + \lambda \frac{dX}{dt} + \mu \frac{d^2 X}{dt^2}.$$

Example(s): 4.1.3 Look to the circular helix, as introduced in Example 4.1.1. An equation of the osculation plane, in the point X, is given by

$$h x^1 \sin t - h x^2 \cos t + a x^3 - a h t = 0.$$

Note that this point passes through the point $x^1 = 0$, $x^2 = 0$, $x^3 = h t$.

4.1.2 The Frenet formulas

In the literature these formulas are also known as the Frenet-Serret formulas, or Serret-Frenet formulas.

Let K be a space curve, parametrised by the arclength s. Let X be a fixed point at K but it is no inflection point. The tangent vector $\dot{X}$ is a unit vector, which in the future will be called $\imath$, so $\dot{X} = \imath$. A straight line through the point X, perpendicular to the tangent line in X, is called normal. The normals in X form a plane, the normal plane in X. The normal in X, which lies in the osculation plane is called the principal normal. The normal, perpendicular to the principal normal, is called the binormal. The plane spanned by the tangent vector and the binormal is called the rectifying plane. Because of the fact that a unit vector does not change of length, holds that $(\dot{X}, \ddot{X}) = 0$, so $\ddot{X}$ stays perpendicular to $\imath$, but it's length has naturally not to be equal to 1. The vector i

is called the curvature vector. We introduce the vectors **n** and **b**, both are unit vectors on the straight lines of the principal normal and the binormal. We agree that **n** points in the direction of $\ddot{X}$ and that **b** points in the direction of $\dot{X} \times \ddot{X}$. The vectors $\imath$, **n** and **b** are oriented on such a way that

$$\mathbf{b} = \imath \times \mathbf{n}, \mathbf{n} = \mathbf{b} \times \imath, \imath = \mathbf{n} \times \mathbf{b}.$$

Let Y be a point in the neighbourhood of X at the space curve K. Let $\triangle\phi$ be the angle between the tangent lines in X and Y and let $\triangle\psi$ be the angle between the binormals in X and Y. Note that $\triangle\psi$ is also the angle between the osculation planes in X and Y.

Definition 4.1.4 The curvature ρ and the torsion τ of the space curve K in the point X is defined by

$$\rho^2 = \left(\frac{d\phi}{ds}\right)^2 = \lim_{\triangle s \to 0} \left(\frac{\triangle\phi}{\triangle s}\right)^2, \tag{4.4}$$

$$\tau^2 = \left(\frac{d\psi}{ds}\right)^2 = \lim_{\triangle s \to 0} \left(\frac{\triangle\psi}{\triangle s}\right)^2. \tag{4.5}$$

We assume that $\rho > 0$. The sign of τ will be determined later on. □

Lemma 4.1.1 There holds that $\rho^2 = (\dot{\imath}, \dot{\imath})$ and $\tau^2 = \left(\dot{\mathbf{b}}, \dot{\mathbf{b}}\right)$. □

Proof Add in a neighbourhood of X to every point of the curve an unit vector **a**, such that the map $s \mapsto \mathbf{a}(s)$ is sufficiently enough differentiable. The length of **a** is equal to 1, there holds that $(\mathbf{a}, \mathbf{a}) = 1$ and there follows that $(\mathbf{a}, \dot{\mathbf{a}}) = 0$. Differentiate this last equation to s and there follows that $(\dot{\mathbf{a}}, \dot{\mathbf{a}}) + (\mathbf{a}, \ddot{\mathbf{a}}) = 0$. Let $\triangle\alpha$ be the angle between $\mathbf{a}(s)$ and $\mathbf{a}(s + \triangle s)$, where $X(s + \triangle s)$ is point in the neighbourhood of $X(s)$. There holds that

$$\cos(\triangle\alpha) = (\mathbf{a}(s), \mathbf{a}(s + \triangle s)).$$

A simple goniometric formula and a Taylor expansion gives as result that

$$1 - 2\sin^2\left(\frac{1}{2}\triangle\alpha\right) = (\mathbf{a}(s), \mathbf{a}(s) + (\triangle s)\,\dot{\mathbf{a}}(s) + \frac{1}{2}(\triangle s)^2\,\ddot{\mathbf{a}}(s) + \mathcal{O}((\triangle s)^3)),\ \triangle s \to 0,$$

so

$$\frac{4\sin^2\left(\frac{1}{2}\triangle\alpha\right)}{(\triangle s)^2} = -(\mathbf{a}(s), \ddot{\mathbf{a}}(s)),\ \triangle s \to 0.$$

Note that $\frac{\sin x}{x} = 1$, for $x \to 0$, such that

$$\lim_{\Delta s \to 0} \left(\frac{\Delta \alpha}{\Delta s} \right)^2 = (\dot{\mathbf{a}}, \dot{\mathbf{a}}).$$

Choose for **a** successively $\imath$ and **b**, and the statement follows. □

The curvature of a curve is a measure for the change of direction of the tangent line. $R = \dfrac{1}{\rho}$ is called the radius of curvature. So far we have confined ourselves till points at K which are no inflection points. But it is easy to define the curvature in an inflection point. In an inflection point is $\ddot{X} = 0$, such that with Lemma 4.1.1 follows that $\rho = 0$. The reverse is also true, the curvature in a point is equal to zero if and only if that point is an inflection point.
For the torsion we have a similar geometrical characterisation. The torsion is zero if and only if the curve belongs to a fixed plane. The torsion measures the speed of rotation of the binormal vector at some point of the curve.
The three vectors $\imath$, **n** and **b** form in each point a orthonormal basis. The consequence is that the derivative of each of these vectors is a linear combination of the other two. These relations we describe in the following theorem. They are called the formules of Frenet.

Theorem 4.1.1 The formules of Frenet read

$$\dot{\imath} = \rho\, \mathbf{n}, \tag{4.6}$$

$$\dot{\mathbf{n}} = -\rho\, \imath + \tau\, \mathbf{b}, \tag{4.7}$$

$$\dot{\mathbf{b}} = -\,\tau\, \mathbf{n}, \tag{4.8}$$

The sign of τ is now also defined. The sign of τ has to be taken so that Equation 4.8 is satisfied. □

Proof The definition of $\dot{\imath}$ is such that it is a multiple of **n**. Out of Lemma 4.1.1 it follows that the length of $\dot{\imath}$ is equal to ρ and so there follows directly Equation 4.6.
With the result of above we conclude that $(\dot{\imath}, \mathbf{b}) = 0$. The fact that $(\mathbf{b}, \imath) = 0$ there follows that $(\dot{\mathbf{b}}, \imath) = -(\mathbf{b}, \dot{\imath}) = 0$. Hereby follows that $\dot{\mathbf{b}}$ is a multiple of **n**. Out of Lemma 4.1.1 follows that $|\tau|$ is the length of **b**. Because of the agreement about the sign of τ, we have Equation 4.8.
Because $(\mathbf{n}, \imath) = 0$ there follows that $(\dot{\mathbf{n}}, \imath) = -(\mathbf{n}, \dot{\imath}) = -\rho$ and because $(\mathbf{n}, \mathbf{b}) = 0$ there follows that $(\dot{\mathbf{n}}, \mathbf{b}) = -(\mathbf{n}, \dot{\mathbf{b}}) = \tau$, such that

$$\dot{\mathbf{n}} = (\dot{\mathbf{n}}, \imath)\, \imath + (\dot{\mathbf{n}}, \mathbf{b})\, \mathbf{b} = -\rho\, \imath + \tau\, \mathbf{b},$$

and Equation 4.7 is proved. □

They call the positive oriented basis $\{\imath, \mathbf{n}, \mathbf{b}\}$ the Frenet frame or also the Frenet trihedron, the repÃÍre mobile, and the moving frame. Build the of the arclength depend matrix

$$F = (\imath, \mathbf{n}, \mathbf{b}\},$$

then holds that $F^T F = I$ and $\det = 1$, so the matrix F is <u>direct orthogonal</u> (so, orthogonal and $\det F = 1$). The formulas of Frenet can now be written as

$$\frac{\mathrm{d}}{\mathrm{d}s} F = F R, \text{ with } R = \begin{pmatrix} 0 & -\rho & 0 \\ \rho & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{4.9}$$

Theorem 4.1.2 Two curves with the same curvature and torsion as functions of the arclength are identical except for position and orientation in space. With a translation and a rigid rotation one of the curves can be moved to coincide with the other. The equations $\rho = \rho(s)$ and $\tau = \tau(s)$ are called the <u>natural equations</u> of the curve. □

Proof Let the given functions ρ and τ be continuous functions of $s \in [0, a)$, with a some positive constant. To prove that there exists a curve K of which the curvature and the torsion are given by respectively ρ and τ. But also to prove that this curve K is uniquely determined apart from a translation and a rigid rotation. The equations 4.9 can be interpreted as a linear coupled system of 9 ordinary differential equations. With the existence and uniqueness results out of the theory of ordinary differential equations follows that there exists just one continuous differentiable solution $F(s)$ of differential equation 4.9, to some initial condition $F(0)$. This matrix $F(0)$ is naturally a direct orthonormal matrix. The question is wether $F(s)$ is for all $s \in [0, a)$ a direct orthonormal matrix? Out of $\dot{F} = F R$ follows that $\dot{F}^T = R^T F^T = -R F^T$. There holds that $\dot{F} F^T = F R F^T$ and $F \dot{F}^T = -F R F^T$, that means that $\frac{\mathrm{d}}{\mathrm{d}s}\left(F F^T\right) = 0$. The matrix $F(s) F^T(s)$ is constant and has to be equal to $F(0) F^T(0) = I$. Out of the continuity of $F(s)$ follows that $\det F(s) = \det F(0) = 1$. So the matrix $F(s)$ is indeed for every $s \in [0, a)$ a direct orthonormal matrix. The matrix $F(s)$ gives the vectors $\imath$, $\mathbf{n}$ and $\mathbf{b}$, from which the searched curve follows.
The arclength parametrisation of this curve is given by

$$X(s) = \mathbf{a} + \int_0^s \imath \, \mathrm{d}s,$$

where $\mathbf{a}$ is an arbitrary chosen vector. Out of this follows directly the freedom of translation of the curve.
Let $\tilde{F}(0)$ be another initial condition and let $\tilde{F}(s)$ be the associated solution. Because of the fact that columns of $\tilde{F}(0)$ form an orthonormal basis, there exist a rigid rotation to transform $\tilde{F}(0)$ into $F(0)$. So there exists a constant direct orthogonal matrix S, such that $\tilde{F}(0) = S F(0)$. The associated solution is given by $\tilde{F}(s) = S F(s)$, because

$$\frac{\mathrm{d}}{\mathrm{d}s}(S F(s)) = S \frac{\mathrm{d}}{\mathrm{d}s} F(s) = S F(s) R.$$

Based on the uniqueness, we conclude that $\tilde{F}(s)$ can be found from the solution $F(s)$ and a rigid rotation. $\square$

A space curve is, apart from its place in the space, completely determined by the functions $\rho(s)$ and $\tau(s)$. They characterize the space curve. This means that all properties of the curve, as far they are independent of its place in the space, can be expressed through relations in ρ and τ. We shall give some characterisation of curves with the help of the curvature and the torsion. Note that out of the equations of Frenet follows that

$$\rho\,\dot{\mathbf{b}} + \tau\,\dot{\imath} = 0. \tag{4.10}$$

We treat the following cases:

1. $\tau = 0$.
   Out of Equation 4.8 follows that $\dot{\mathbf{b}} = 0$, such that
   $$\frac{\mathrm{d}}{\mathrm{d}s}(X(s), \mathbf{b}) = (\dot{X}, \mathbf{b}) + (X, \dot{\mathbf{b}}) = (\imath, \mathbf{b}) = 0.$$
   Evidently is $(X(s), \mathbf{b})$ a constant, say α. Than holds that $X(s) - \alpha\,\mathbf{b}$ for every s lies in a plane perpendicular to $\mathbf{b}$. The space curve lies in a fixed plane.

2. $\rho = 0$.
   Out of Equation 4.6 follows that $\dot{\imath} = 0$, such that
   $$X(s) = \mathbf{a} + s\,\imath(0).$$
   So the space curve is a straight line.

3. ρ =constant and nonzero, $\tau = 0$.
   We know already that the space curve lies in a fixed plane. There holds that
   $$\frac{\mathrm{d}^3}{\mathrm{d}s^3}X = \ddot{\imath} = \frac{\mathrm{d}}{\mathrm{d}s}(\rho\,\mathbf{n}) = \rho\,\dot{\mathbf{n}} = -\rho^2\,\imath,$$
   that means that
   $$\frac{1}{\rho^2}\frac{\mathrm{d}^3}{\mathrm{d}s^3}X + \dot{X} = 0,$$
   such that $\rho^{-2}\,\ddot{X} + X = \mathbf{m}$, with $\mathbf{m}$ a constant vector. There follows that
   $$|X - \mathbf{m}| = \frac{1}{\rho^2}\,|\ddot{X}| = \frac{1}{\rho}|\mathbf{n}| = \frac{1}{\rho},$$
   so the space curve is a circle with center $\mathbf{m}$ and radius ρ^{-1}.

4. $\dfrac{\tau}{\rho}$ =constant, with ρ and τ both nonzero.
   Out of Equation 4.10 follows that $\mathbf{b} + \dfrac{\tau}{\rho}\,\imath$ is constant, let say to $\mathbf{u}$. Note that $(\mathbf{u}, \mathbf{u}) = 1 + \dfrac{\tau^2}{\rho^2}$ and $(\mathbf{u}, \mathbf{b}) = 1$ such that out of

$$\dot{X} - \frac{\rho}{\tau}\mathbf{u} = -\frac{\rho}{\tau}\mathbf{b}$$

follows that

$$\left(\dot{X} - \frac{\rho}{\tau}\mathbf{u},\, \mathbf{u}\right) = -\frac{\rho}{\tau} = -\frac{\rho}{\tau}\left(1 + \frac{\tau^2}{\rho^2}\right)^{-1}(\mathbf{u},\, \mathbf{u}), \tag{4.11}$$

so

$$\left(\dot{X} - \frac{\rho\,\tau}{\rho^2 + \tau^2}\,\mathbf{u},\, \mathbf{u}\right) = 0.$$

Evidently is $\left(X - \frac{\rho\,\tau}{\rho^2+\tau^2}\, s\,\mathbf{u},\, \mathbf{u}\right)$ a constant and equal to $(X(0), u)$. The vector $X - \frac{\rho\,\tau}{\rho^2+\tau^2}\, s\,\mathbf{u}$ lies for every s in a plane perpendicular to $\mathbf{u}$ and through $X(0)$. We conclude that the space curve is a cylindrical helix.

Notice(s): 4.1.1

- The tangent vector $\dot{X}$ makes a constant angle with some fixed vector $\mathbf{u}$, see Equation 4.11.
- The function $h(s) = (X(s) - X(0),\, \mathbf{u})$ tells how $X(s)$ has "risen"in the direction $\mathbf{u}$, since leaving $X(0)$. And $\frac{\mathrm{d}h}{\mathrm{d}s} = \left(\dot{X},\, \mathbf{u}\right)$ is constant, so $h(s)$ rises at a constant rate relative to the arclength.

5. ρ = constant and nonzero, τ = constant and nonzero.
We know already that the space curve is a cylindrical helix. There holds that

$$\frac{\mathrm{d}^3}{\mathrm{d}s^3}X = \ddot{\imath} = -\rho^2\,\imath + \rho\,\tau\,\mathbf{b} = -\rho^2\,\imath + \rho\,\tau\,\mathbf{u} - \tau^2\,\imath,$$

where $\mathbf{u}$ is the same constant vector as used in the previous example. There follows that

$$\frac{1}{\rho^2 + \tau^2}\frac{\mathrm{d}^3}{\mathrm{d}s^3}X + \dot{X} - \frac{\rho\,\tau}{\rho^2 + \tau^2}\,\mathbf{u} = 0.$$

Evidently is the vector $\frac{1}{\rho^2 + \tau^2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}X + X - \frac{\rho\,\tau}{\rho^2 + \tau^2}\,s\,\mathbf{u}$ constant and equal to the constant vector $\mathbf{m}$, with $\mathbf{m} = \frac{\rho}{\rho^2 + \tau^2}\,\mathbf{n}(0) + X(0)$. We conclude that

$$X(s) - \frac{\rho\,\tau}{\rho^2 + \tau^2}\,s\,\mathbf{u} - \mathbf{m} = -\frac{\rho}{\rho^2 + \tau^2}\,\mathbf{n}(s),$$

such that

$$\left|X(s) - \frac{\rho\,\tau}{\rho^2 + \tau^2}\,s\,\mathbf{u} - \mathbf{m}\right| = \frac{\rho}{\rho^2 + \tau^2}.$$

The space curve is, as we already know, a cylindrical helix, especially a circular helix.

Section 4.2 Differential geometry of surfaces in $\mathbb{R}^3$

4.2.1 Surfaces

We consider again the standard basis $\{E_i\}$ of $\mathbb{R}^3$, with which every point $X \in \mathbb{R}^3$ can be written as $X = x^i E_i$. Let x^i be real functions of the two real parameters u^1 and u^2, with $(u^1, u^2) \in \mathbb{R} \subset \Omega\mathbb{R}^2$ and Ω open. We suppose that that the functions x^i are enough times differentiable to both variables, such that in the remainder there will be no difficulties with respect to differentiation. Further we assume that the matrix, formed by the partial derivatives $\dfrac{\partial x^i}{\partial u^j}$, has rank two. This means that the vectors $\partial_1 X$ and $\partial_2 X$ are linear independent in every point X. (We use in this section the notation ∂_j for $\dfrac{\partial}{\partial u^j}$).

Definition 4.2.1 A surface S in $\mathbb{R}^3$ is the set of points $X = X(u^1, u^2) = x^i(u^1, u^2)\, E_i$ with $(u^1, u^2) \in \Omega$. □

We call the functions $x^i = x^i(u^j)$ a parametric representation of the surface S. The parameters u^j are the coordinates at the surface. If one of these coordinates is kept constant, there is described a curve at the surface. Such a curve belongs to the parametric representation and is called a parametric curve. The condition that the rank of the matrix $\left[\partial_j x^i\right]$ is equal to two expresses the fact that the two parametric curves u_1 =constant and u_2 =constant can not fall together. Just as the curves in $\mathbb{R}^3$, there are infinitely many possibilities to describe the same surface S with the help of a parametric representation. With the help of the substitution $u^1 = u^1(u^{1'}, u^{2'})$, $u^2 = u^2(u^{1'}, u^{2'})$, where we suppose that the determinant of the matrix $\left[\dfrac{\partial u^i}{\partial u^{i'}}\right]$ is nonzero, there is obtained a new parametric representation of the surface S, with the coordinates $u^{i'}$. The assumption that $\det\left[\dfrac{\partial u^i}{\partial u^{i'}}\right] \neq 0$ is again guaranteed by the fact that the rank of the matrix $\left[\partial_j x^i\right]$ is equal to two. From now on the notation of the partial derivatives $\dfrac{\partial u^i}{\partial u^{i'}}$ will be $A^i_{i'}$.

Let S be a surface in $\mathbb{R}^3$ with the coordinates u^i. A curve K at the surface S can be described by $u^i = u^i(t)$, where t is a parameter for K. The tangent vector in a point X at this curve is given by

$$\frac{dX}{dt} = \frac{du^i}{dt} \partial_i X,$$

which is a combination of the vectors $\partial_1 X$ and $\partial_2 X$. The tangent lines in X to all the curves through X at the surface lie in a plane. This plane is the tangent plane in X to S, notation $T_X(S)$. This tangent plane is a 2−dimensional linear subspace of the tangent space $T_X(\mathbb{R}^3)$. The vectors $\partial_1 X$ and $\partial_2 X$ form on a natural way a basis of this subspace. With the transition to other coordinates $u^{i'}$ holds $\partial_{i'} X = A^i_{i'} \partial_i X$, this means that in the tangent plane there is a transistion to another basis.

Example(s): 4.2.1 A sphere with radius R can be described by the parametric representation

$$x^1(\theta, \phi) = R \sin\theta \cos\phi,\ x^2(\theta, \phi) = R \sin\theta \sin\phi,\ x^3(\theta, \phi) = R \cos\theta,$$

with $0 < \theta < \pi$ and $0 < \phi < 2\pi$. Note that the rank of the matrix formed by the columns $\partial_\theta X$ and $\partial_\phi X$ is equal to two, if $\theta \neq 0$ and $\theta \neq \pi$.

4.2.2 The first fundamental tensor field

Let S be a surface, parametrised with the coordinates u^1 and u^2. In every point $X \in S$ we notate the tangent plane in X to S by $T_X(S)$. This space is a two-dimensional subspace of the tangent space $T_X(\mathbb{R}^3)$ in X. The basis of this space is $\{\partial_i X\}$, which is determined by the parametrisation $X = X(u^i)$. Let $(\cdot, \cdot)$ be the Euclidean inner product at $\mathbb{R}^3$. This inner product can be transferred on a natural way to the tangent space $T_X(\mathbb{R}^3)$. The obtained inner product at $T_X(\mathbb{R}^3)$ we notate by $(\cdot, \cdot)_X$, see also Subsection 3.4.2. This innerproduct is naturally also an inner product at the subspace $T_X(S)$.

Definition 4.2.2 The first fundamental tensor field is the fundamental tensor field that adds the inner product to $T_X(S)$. Out of convenience, we notate the first fundamental tensor field by g. □

The components of the first fundamental tensor field, belonging to the coordinates u^i, are given by

$$g_{ij} = (\partial_i X, \partial_j X)_X,$$

such that g can be written as $g = g_{ij}\, du^i \otimes du^j$. Hereby is $\{du^1, du^2\}$ the reciprocal basis, which belongs to the basis $\{\partial_1 X, \partial_2 X\}$.

The vectors $\partial_1 X$ and $\partial_2 X$ are tangent vectors to the parameter curves $u^2 = C$ and $u^1 = C$. If the parameters curves intersect each other at an angle α, than holds that

$$\cos\alpha = \frac{(\partial_1 X, \partial_2 X)_X}{|\partial_1 X|\,|\partial_2 X|} = \frac{g_{12}}{\sqrt{g_{11}\,g_{22}}}.$$

It is evident that the parameter curves intersect each other perpendicular, if $g_{12} = 0$. A parametrisation u^i of a surface S is called orthogonal if $g_{12} = 0$.

Example(s): 4.2.2 The in Example 4.2.1 given parametrisation of a sphere with radius R is orthogonal. Because there holds that $g_{11} = R^2$, $g_{22} = R^2 \sin^2\theta$ and $g_{12} = g_{21} = 0$.

4.2.3 The second fundamental tensor field

Let S be a surface in $\mathbb{R}^3$ with parametrisation $x^i = x^i(u^j)$. In a point X are the vectors $\partial_j X$ tangent vectors to the parameter curves through X. They form a basis of the tangent plane $T_X(S)$. Let $\mathbf{N}_X$ be the vector in X perpendicular to $T_X(S)$, with length 1 and it points in the direction of $\partial_1 X \times \partial_2 X$. A basis of $T_X(\mathbb{R}^3)$ is formed by the vectors $\partial_1 X \times \partial_2 X$ and $\mathbf{N}_X$.
The derivatives $\partial_i\partial_j X$ are linear combinations of the vectors $\partial_1 X$, $\partial_2 X$ and $\mathbf{N}_X$. We note this as follows

$$\partial_i\partial_j X = \left\{ {k \atop i \quad j} \right\} \partial_k X + h_{ij}\,\mathbf{N}_X. \tag{4.12}$$

The notation $\left\{ {k \atop i \quad j} \right\}$ indicates an analogy with the in Subsection 3.6.3 introduced Christoffel symbols. Later this will become clear.
Out of the representation 4.12 follows

$$(\partial_i\partial_j X, \partial_l X) = \left\{ {m \atop i \quad j} \right\} g_{ml} + h_{ij}\,(\mathbf{N}_X, \partial_l X) = \left\{ {m \atop i \quad j} \right\} g_{ml},$$

such that

$$\left\{ {k \atop i \quad j} \right\} = g^{kl}\,(\partial_i\partial_j X, \partial_l X).$$

It is clear that

$$h_{ij} = (\partial_i\partial_j X, \mathbf{N}_X). \tag{4.13}$$

Lemma 4.2.1 The Christoffel symbols are not the components of a $\binom{1}{2}$-tensor field, the functions h_{ij} are the components of a covariant 2-tensor field.

Proof Let $x^i(u^{j'})$ be a second parametric representation of the surface S. There holds that

$$\left\{\begin{matrix} & k' & \\ i' & & j' \end{matrix}\right\} = g^{k'l'}\,(\partial_{i'}\partial_{j'}X, \partial_{l'}X) = A^{k'}_{k}\,A^{j}_{j'}\,A^{i}_{i'}\left\{\begin{matrix} & k & \\ i & & j \end{matrix}\right\} + A^{k'}_{k}\,(\partial_{i'}A^{k}_{j'}).$$

The second term is in general not equal to zero, so the Christoffel symbols are not the components of a tensor field. (See also Formula 3.8, the difference is the inner product!) Furthermore holds that

$$\begin{aligned} h_{i'j'} &= (\partial_{i'}\partial_{j'}X, \mathbf{N}_X) = (\partial_{i'}(A^{j}_{j'}\,\partial_j X), \mathbf{N}_X) = ((\partial_{i'}A^{j'}_{j})\partial_j X + A^{j}_{j'}\,(\partial_{i'}\partial_j X), \mathbf{N}_X) \\ &= A^{j}_{j'}\,(\partial_{i'}\partial_j X, \mathbf{N}_X) = A^{j}_{j'}\,A^{i}_{i'}\,(\partial_i\partial_j X, \mathbf{N}_X) = A^{j}_{j'}\,A^{i}_{i'}\,h_{ij}, \end{aligned}$$

from which follows that the functions h_{ij} are the components of a tensor field. □

Definition 4.2.3 The <u>second fundamental tensor field</u> is the covariant 2-tensor field of which the components, with respect to the base u^k, are given by h_{ij}, so $h = h_{ij}\,\mathrm{d}u^i \otimes \mathrm{d}u^j$. □

Lemma 4.2.2 The Christoffel symbols are completely described with the help of the components of the first fundamental tensor field. There holds that

$$\left\{\begin{matrix} & k & \\ i & & j \end{matrix}\right\} = \frac{1}{2}\,g^{kl}\,(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}). \tag{4.14}$$

Proof There holds that

$$\begin{aligned} \partial_i g_{jl} &= \partial_i(\partial_j X, \partial_l X) = (\partial i\partial_j X, \partial_l X) + (\partial_j X, \partial i\partial_l X), \\ \partial_j g_{li} &= \partial_j(\partial_l X, \partial_i X) = (\partial j\partial_l X, \partial_i X) + (\partial_l X, \partial j\partial_i X), \\ \partial_l g_{ij} &= \partial_l(\partial_i X, \partial_j X) = (\partial l\partial_i X, \partial_j X) + (\partial_i X, \partial l\partial_j X), \end{aligned}$$

such that

$$\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij} = 2\,(\partial i\partial_j X, \partial_l X) = 2\,g_{lk}\left\{\begin{matrix} & k & \\ i & & j \end{matrix}\right\},$$

from which Formula 4.14 easily follows. □

Note that Formula 4.14 corresponds with Formula 3.9.

Theorem 4.2.1 The intersection of a surface S with a flat plane, that lies in some small neighbourhood of a point X at S and is parallel to $T_X(S)$, is in the first approximation a hyperbola, ellipse or a pair of parallel lines and is completely determined by the second fundamental tensor. □

Proof We take Cartesian coordinates x, y and z in $\mathbb{R}^3$ such that X is the origin and the tangent plane $T_X(S)$ coincides with the plane $z = 0$. In a sufficiently small neighbourhood of the origin, the surface S can be descibed by an equation of the form $z = f(x, y)$. A parametric representation of S is given by $x^1 = x$, $x^2 = y$ and $x^3 = z = f(x, y)$. We assume that the function f is enough times differentiable, such that in a neighbourhood of the origin the equation of S can written as

$$
\begin{aligned}
z &= f(x, y) = f(0,0) + \frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial y}(0,0) + \frac{1}{2}(r\,x^2 + 2\,s\,x\,y + t\,y^2) + \text{h.o.t.} \\
&= \frac{1}{2}(r\,x^2 + 2\,s\,x\,y + t\,y^2) + \text{h.o.t.},
\end{aligned}
$$

with

$$
r = \frac{\partial^2 f}{\partial x^2}(0,0),\; r = \frac{\partial^2 f}{\partial x\,\partial y}(0,0) \text{ and } t = \frac{\partial^2 f}{\partial y^2}(0,0).
$$

The abbreviation h.o.t. means higher order terms. A plane that lies close to X and is parallel to the tangent plane to S at X is described by $z = \varepsilon$, with ε small enough. The intersection of this plane with X is given in a first order approximation by the equation

$$
r\,x^2 + 2\,s\,x\,y + t\,y^2 = 2\,\varepsilon.
$$

The tangent vectors to the coordinate curves, with respect to the coordinates x and y, in the origin, are given by

$$
\partial_x X = \left(1,\, 0,\, \frac{\partial f}{\partial x}\right)^T
$$

and

$$
\partial_y X = \left(1,\, 0,\, \frac{\partial f}{\partial y}\right)^T,
$$

such that in the origin holds

$$
\partial_x \partial_x X = (0,\, 0,\, r)^T,\; \partial_x \partial_y X = (0,\, 0,\, s)^T \text{ and } \partial_y \partial_y X = (0,\, 0,\, t)^T.
$$

Furthermore holds in the origin $X = 0$ that $\mathbf{N}_X = (0.\,0\,1)^T$. It is evident that $h_{11} = r$, $h_{12} = h_{21} = s$, $h_{22} = t$, see Formula 4.13, such that the equation of the intersection is given by

$$
h11\,x^2 + 2\,h_{12}\,x\,y + h_{22}\,y^2 = 2\,\varepsilon.
$$

We conclude that the intersection is completely determined by the numbers h_{ij} and that the intersection is an ellipse if $\det[h_i j] > 0$, a hyperbola if $\det[h_i j] < 0$ and a pair of parallel lines if $\det[h_i j] = 0$. □

This section will be closed with a handy formula to calculate the components of the second fundamental tensor field. Note that $\partial_1 X \times \partial_1 X = \lambda\, \mathbf{N}_X$, with $\lambda = |\partial_1 X \times \partial_2 X|$. This λ can be represented with components of the first fundamental tensor field. There holds

$$\lambda = |\partial_1 X|\, |\partial_2 X|\, sin\vartheta,$$

with ϑ the angle between $\partial_1 X$ and $\partial_2 X$, such that $0 < \vartheta < \pi$. There follows that

$$\lambda = \sqrt{g_{11}\, g_{22}\, (1 - \cos^2\vartheta)} = \sqrt{g_{11}\, g_{22}\, (1 - \frac{g_{12}^2}{g_{11}\, g_{22}})} = \sqrt{g_{11}\, g_{22} - g_{12}^2} = \sqrt{\det[g_{ij}]}.$$

Furthermore is

$$h_{ij} = (\partial_i \partial_j X, \mathbf{N}_X) = \frac{1}{\lambda}(\partial_1 X \times \partial_2 X,\, \partial_i \partial_j X) = \frac{1}{\sqrt{\det[g_{ij}]}} \det(\partial_1 X,\, \partial_2 X,\, \partial_i \partial_j X).$$

4.2.4 Curves at a surface

Let S be a surface in $\mathbb{R}^3$ with the parametric representation $x^i = x^i(u^j)$. Furthermore is K a curve at S, parametrised by its arclength s, so $u^j = u^j(s)$, such that $x^i = x^i(u^j(s))$. We notated the differentation of u^j to s, just as done earlier, with $\dot{u}^j$. The unit tangent vector at K in a point X is the vector $\imath$, which lies in the tangent plane $T_X(S)$. There holds

$$\imath = \dot{X} = \dot{u}^j\, \partial_j X.$$

The curvature vector $\ddot{X}$ is the vector along the principal normal of the curve K and satisfies

$$\begin{aligned} \ddot{X} = \dot{\imath} &= \ddot{u}^j\, \partial_j X + \dot{u}^j\, \dot{u}^k\, \partial_k \partial_j X = \ddot{u}^j\, \partial_j X + \dot{u}^j\, \dot{u}^k \left(\left\{ \begin{matrix} & l & \\ k & & j \end{matrix} \right\} \partial_l X + h_{kj}\, \mathbf{N}_X \right) \\ &= \left(\ddot{u}^l + \dot{u}^j\, \dot{u}^k \left\{ \begin{matrix} & l & \\ j & & k \end{matrix} \right\} \right) \partial_l X + \dot{u}^j\, \dot{u}^k\, h_{jk}\, \mathbf{N}_X. \end{aligned} \tag{4.15}$$

The length of the curvature vector is given by the curvature ρ in X (see Lemma 4.1.1).

Definition 4.2.4 The geodesic curvature of K is the length of the projection of $\ddot{X}$ onto the tangent plane $T_X(S)$. □

It is, with the help of Formula 4.15, easy to see that the geodesic curvature can be calulated with the help of the formula

$$\sqrt{\left(\ddot{u}^i + \begin{Bmatrix} i \\ l \quad k \end{Bmatrix} \dot{u}^l \, \dot{u}^k\right) \left(\ddot{u}^j + \begin{Bmatrix} j \\ p \quad q \end{Bmatrix} \dot{u}^p \, \dot{u}^q\right) g_{ij}}. \tag{4.16}$$

Note that the geodesic curvature only depends on components of the first fundamental tensor field.

Definition 4.2.5 The principal curvature of K in a point X is the length of $\ddot{X}$ at $\mathbf{N}_X$. □

Out of Formula 4.15 follows that the principal curvature is given by $\dot{u}^j \, \dot{u}^k \, h_{jk}$. Note that the principal curvature only depends on the components of the second fundamental tensor field and the values of $\dot{u}^i$. These last values determine the direction of $\imath$. This means that different curves at the surface S, with the same tangent vector in a point X at S, have an equal principal curvature. This result is known as the theorem of Meusnier

Definition 4.2.6 A geodesic line or geodesic of the surface S is a curve at S, of which in every point the principal normal and the normal at the surface fall together. □

Out of Formula 4.15 follows that a geodesic is decribed by the equations

$$\ddot{u}^i + \begin{Bmatrix} i \\ j \quad k \end{Bmatrix} \dot{u}^j \, \dot{u}^k = 0. \tag{4.17}$$

This is a non-linear inhomogeneous coupled system of two ordinary differential equations of the second order in u^1 and u^2. An analytic solution is most of the time difficult or not at all to determine. Out of the theory of ordinary differential equations follows that there is exactly one geodesic line through a given point and a given direction.
If a curve is not parametrised with its arclength, but with another parameter, for instance t, then is the arclength given by Formula 4.1. This formula can be expressed by the coordinates u^j and the components of the first fundamental tensor field. It is easy to see that the arclength s of a curve K at S, with parameter t, between the points $t = t_0$ and $t = t_1$ is given by

$$s = \int_{t_0}^{t_1} \sqrt{g_{ij} \frac{\mathrm{d}u^i}{\mathrm{d}t} \frac{\mathrm{d}u^j}{\mathrm{d}t}} \, \mathrm{d}t \tag{4.18}$$

We use this formula to prove the following theorem.

Theorem 4.2.2 Let X_0 and X_1 be given points at S and K a curve at S through X_0 and X_1. If the curve K has minimal length than is K a geodesic. □

Proof Let K have minimal length and parametrised by its arclength s,
with $s_0 \leq s \leq s_1$. So $X_0 = x^i(u^j(s_0))\,E_i$ and $X_1 = x^i(u^j(s_1))\,E_i$. Define de function T by

$$T(u^k, \dot{u}^k) = \sqrt{g_{ij}(u^k)\,\dot{u}^i\,\dot{u}^j}.$$

Note that T has the value 1 in every point of K. We vary now K on a differentiable way on the surface S, where we keep the points X_0 and X_1 fixed. So we obtain a new curve $\tilde{K}$. This curve $\tilde{K}$ can be represented by $u^j(s) + \varepsilon\,\eta^j(s)$, with η^j differentiable and $\eta^j(s_0) = \eta^j(s_1) = 0$. The parameter s is not necessarily the arclength parameter of $\tilde{K}$. Consequently the length of $\tilde{K}$ is given by

$$\int_{s_0}^{s_1} \sqrt{g_{ij}(u^k + \varepsilon\,\eta^k)\,\frac{\mathrm{d}(u^i + \varepsilon\,\eta^i)}{\mathrm{d}s}\,\frac{\mathrm{d}(u^j + \varepsilon\,\eta^j}{\mathrm{d}s})}\mathrm{d}s = \int_{s_0}^{s_1} T(u^k + \varepsilon\,\eta^k, \dot{u}^k + \varepsilon\,\dot{\eta}^k)\,\mathrm{d}s.$$

Because the length of K is minimal, the expression above has it's minimum for $\varepsilon = 0$, that means that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\int_{s_0}^{s_1} T(u^k + \varepsilon\,\eta^k, \dot{u}^k + \varepsilon\,\dot{\eta}^k)\,\mathrm{d}s\right)_{\varepsilon = 0} = 0.$$

Out of this follows

$$\int_{s_0}^{s_1} \left(\eta^k \frac{\partial T}{\partial u^k} + \dot{\eta}^k \frac{\partial T}{\partial \dot{u}^k}\right)\mathrm{d}s = \int_{s_0}^{s_1} \eta^k \left(\frac{\partial T}{\partial u^k} - \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial T}{\partial \dot{u}^k}\right)\mathrm{d}s = 0, \tag{4.19}$$

hereby is used partial integration and there is used that $\eta^k(s_0) = \eta^k(s_1) = 0$. Because Formula 4.19 should apply to every function η^k, we find that

$$\frac{\partial T}{\partial u^k} - \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial T}{\partial \dot{u}^k} = 0. \tag{4.20}$$

Because of the fact that T in every point of K takes the value 1, it is no problem to replace T by T^2 in Formula 4.20 and the equations become

$$\frac{\partial}{\partial u^k}\left(g_{ij}\,\dot{u}^i\,\dot{u}^j\right) - \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial}{\partial \dot{u}^k}\left(g_{ij}\,\dot{u}^i\,\dot{u}^j\right) = 0,$$

or

$$\dot{u}^i\,\dot{u}^j\,\partial_k g_{ij} - \frac{\mathrm{d}}{\mathrm{d}s}\left(g_{ki}\,\dot{u}^i + g_{kj}\,\dot{u}^j\right) = 0,$$

or

$$\dot{u}^i\,\dot{u}^j\,\partial_k g_{ij} - \left(2\ddot{u}^i\,g_{ki} + (\partial_j g_{ki} + \partial_i g_{kj})\,\dot{u}^i\,\dot{u}^j\right) = 0,$$

or

$$2\,g_{ki}\,\ddot{u}^i + (\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij})\,\dot{u}^i\,\dot{u}^j = 0,$$

or

$$\ddot{u}^i + \left\{ \begin{matrix} k \\ i \quad j \end{matrix} \right\} \dot{u}^i \, \dot{u}^j = 0.$$

This are exactly the equations for geodesic lines. Because of the fact that K satisfies these equations, is K a geodesic through X_0 and X_1. □

4.2.5 The covariant derivative at surfaces

Let S be a surface in $\mathbb{R}^3$ with the parametrisation $x^i = x^i(u^j)$ and let K be a curve at S with the parametrisation $u^i = u^j(t)$. Furthermore is $\mathbf{v}(t)$ for every t a vector in $T_{X(t)}(S)$. Such a vector field $\mathbf{v}$ is called a tangent vector field to S, defined at the points of the curve K. The vector $\mathbf{v}$ can also be written as

$$\mathbf{v} = v^i \, \partial_i X \, (u^j(t)).$$

Example(s): 4.2.3 The vector field formed by the tangent vector to K is a tangent vector both to S and K. This tangent vector field has contravariant components $\dfrac{\mathrm{d}u^i(t)}{\mathrm{d}t}$. There holds indeed that

$$\frac{\mathrm{d}X(u^j(t))}{\mathrm{d}t} = \frac{\mathrm{d}u^j(t)}{\mathrm{d}t} \, \partial_j X(t).$$

Example(s): 4.2.4 The vector field formed by the basis vectors $\partial_j X(t)$, with i fixed, is a tangent vector field and the contravariant components are δ_i^j.

Example(s): 4.2.5 The vector field formed by the reciproke basis vectors $\mathrm{d}u^i$, with i fixed, is a tangent vector field and has covariant components δ_i^j and contravariant components g^{ij}.

Let $\mathbf{v}$ be tangent vector field. In general, the derivative $\dfrac{\mathrm{d}v(t)}{\mathrm{d}t}$ will not be an element of the tangent plane $T_{X(t)}(S)$. In the following definition we will give an definition of a derivative which has that property.

Definition 4.2.7 The covariant derivative of $\mathbf{v}$ along K, notation $\frac{\nabla \mathbf{v}}{\mathrm{d}t}$, is defined by

$$\frac{\nabla \mathbf{v}}{\mathrm{d}t} = \mathcal{P}_X \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t},$$

with $\mathcal{P}_X$ the projection at the tangent plane $T_X(S)$. □

The covariant differentiation in a point X at S is a linear operation such that the tangent vectors at S, which grasp at the curve K, is imaged at $T_X(S)$. For every scalar field f at K holds

$$\frac{\nabla(f\mathbf{v})}{\mathrm{d}t} = \mathcal{P}_X \frac{\mathrm{d}(f\mathbf{v})}{\mathrm{d}t} = \mathcal{P}_X \left(\frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{v} + f\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \right) = \frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{v} + f\frac{\nabla \mathbf{v}}{\mathrm{d}t}. \tag{4.21}$$

Note that for every $\mathbf{w} \in T_{X(t)}(S)$ holds that

$$\left(\mathbf{w}, \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) = \left(\mathbf{w}, \frac{\nabla \mathbf{v}}{\mathrm{d}t}\right),$$

because $\mathbf{w}$ is perpendicular to $\mathbf{N}_X$. For two tangent vector fields $\mathbf{v}$ and $\mathbf{w}$ alnog the same curve of the surface there follows that

$$\frac{\mathrm{d}(\mathbf{v}, \mathbf{w})}{\mathrm{d}t} = \left(\mathbf{v}, \frac{\nabla \mathbf{w}}{\mathrm{d}t}\right) + \left(\frac{\nabla \mathbf{w}}{\mathrm{d}t}, \mathbf{w}\right). \tag{4.22}$$

This formula is a rule of Leibniz.

Example(s): 4.2.6 Consider the vector field out of Example 4.2.3. Call this vector field $\mathbf{w}$. The covariant derivative of this vector field along the curve K can be expressed by Christoffel symbols. There holds

$$\begin{aligned}
\frac{\nabla \mathbf{w}}{\mathrm{d}t} &= \mathcal{P}_X \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} = \mathcal{P}_X \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}u^j}{\mathrm{d}t} \partial_j X \right) \right) = \mathcal{P}_X \left(\frac{\mathrm{d}^2 u^j}{\mathrm{d}t^2} \partial_j X + \frac{\mathrm{d}u^j}{\mathrm{d}t} \frac{\mathrm{d}}{\mathrm{d}t} \partial_j X \right) \\
&= \mathcal{P}_X \left(\frac{\mathrm{d}^2 u^j}{\mathrm{d}t^2} \partial_j X + \frac{\mathrm{d}u^j}{\mathrm{d}t} \frac{\mathrm{d}u^k}{\mathrm{d}t} \partial_k \partial_j X \right) = \frac{\mathrm{d}^2 u^j}{\mathrm{d}t^2} \partial_j X + \frac{\mathrm{d}u^j}{\mathrm{d}t} \frac{\mathrm{d}u^k}{\mathrm{d}t} \left\{ \begin{matrix} & l & \\ k & & j \end{matrix} \right\} \partial_l X \\
&= \left(\frac{\mathrm{d}^2 u^j}{\mathrm{d}t^2} + \left\{ \begin{matrix} & j & \\ k & & l \end{matrix} \right\} \frac{\mathrm{d}u^k}{\mathrm{d}t} \frac{\mathrm{d}u^l}{\mathrm{d}t} \right) \partial_j X,
\end{aligned}$$

where we made use of Formula 4.12.

Example(s): 4.2.7 Consider the vector field out of Example 4.2.4. Also the covariant derivative of this vector field is to express in Christoffel symbols.

$$\frac{\nabla}{dt}\partial_i X = \frac{\nabla}{dt}\left(\delta_i^j \partial_j X\right) = \mathcal{P}_X\left(\left(\frac{d}{dt}\delta_i^j\right)\partial_j X + \delta_i^j \frac{d}{dt}\partial_j X\right) = \delta_i^j \frac{du^k}{dt}\left\{\begin{matrix} & l & \\ k & & j\end{matrix}\right\}\partial_l X$$
$$= \left\{\begin{matrix} & j & \\ i & & k\end{matrix}\right\}\frac{du^k}{dt}\partial_j X.$$

Example(s): 4.2.8 Consider the vector field out of Example 4.2.5. There holds

$$0 = \frac{d}{dt}\delta_i^j = \frac{d}{dt}\left(du^i, \partial_j X\right) = \left(\frac{\nabla}{dt}du^i, \partial_j X\right) + \left(du^i, \frac{\nabla}{dt}\partial_j X\right),$$

where the rule of Leibniz 4.22 is used. Out of this result follows that

$$\left(\frac{\nabla}{dt}du^i, \partial_j X\right) = -\left(du^i, \frac{\nabla}{dt}\partial_j X\right) = -\left(du^i, \left\{\begin{matrix} & l & \\ j & & k\end{matrix}\right\}\frac{du^k}{dt}\partial_l X\right) = -\left\{\begin{matrix} & i & \\ j & & k\end{matrix}\right\}\frac{du^k}{dt},$$

such that

$$\frac{\nabla}{dt}du^i = -\left\{\begin{matrix} & i & \\ j & & k\end{matrix}\right\}\frac{du^k}{dt}du^j.$$

In particular, we can execute the covariant differentiation along the parameter curves. These are obtained by taking one of the variables u^k as parameter and the other variables u^j, $j \neq k$ fixed. Then follows out of Example 4.2.7 that for the covariant derivative of the basis vectors along the parameter curves that

$$\frac{\nabla}{du^k}\partial_i X = \left\{\begin{matrix} & j & \\ i & & l\end{matrix}\right\}\frac{du^l}{du^k}\partial_j X. \quad (u^k \text{ is the parameter instead of } t.)$$

At the same way follows out Example 4.2.8 that the covariant derivatives of the reciproke basis vectors along the parameter curves are given by

$$\frac{\nabla}{du^l}du^i = -\left\{\begin{matrix} & i & \\ j & & k\end{matrix}\right\}\frac{du^k}{du^l}du^j = -\left\{\begin{matrix} & i & \\ j & & l\end{matrix}\right\}du^j. \tag{4.23}$$

In general the covariant derivative of a tangent vector field $\mathbf{v}$ along a parameter curve is given by

$$\frac{\nabla}{du^k}\mathbf{v} = \frac{\nabla}{du^k}\left(v^j\partial_j X\right) = \partial_k v^j\,\partial_j X + v^j\frac{\nabla}{du^k}\partial_j X = \left(\partial_k v^j + v^l\left\{\begin{matrix} & j & \\ k & & l\end{matrix}\right\}\right)\partial_j X,$$

and here we used Formula 4.21. The covariant derivative of a tangent vector field with respect to the reciproke basis vectors is also easily to write as

$$\frac{\nabla}{du^k}\mathbf{v} = \frac{\nabla}{du^k}\left(v_j\, du^j\right) = \partial_k v_j\, du^j + v_j \frac{\nabla}{du^k} du^j = \left(\partial_k v_j - v_l \left\{ \begin{matrix} & l & \\ k & & j \end{matrix} \right\}\right) du^j. \tag{4.24}$$

Definition 4.2.8 Let $\mathbf{v} = v^j \partial_j X$ be a tangent vector field, defined at the whole surface S, then we define

$$\nabla_k v^j = \partial_k v^j + v^l \left\{ \begin{matrix} & j & \\ k & & l \end{matrix} \right\}. \tag{4.25}$$

□

Lemma 4.2.3 The functions $\nabla_k v^j$, given by Formula 4.25 are the components of a $\binom{1}{1}$-tensor field at S. This tensor field is called the covariant derivative of $\mathbf{v}$ at S.

Proof Prove it yourself. □

Also the functions $\nabla_k v_j$, defined by

$$\nabla_k v_j = \partial_k v_j - v_l \left\{ \begin{matrix} & l & \\ k & & j \end{matrix} \right\}, \tag{4.26}$$

see Formula 4.24, are components of a tensor field, a $\binom{0}{2}$-tensor field.

Evenso we define covariant differentiation at S of 2-tensor fields.

Definition 4.2.9 Let ϕ_{ij}, ϕ^{ij} and ϕ_i^j be the components of respectively a $\binom{0}{2}$-tensor field, a $\binom{2}{0}$-tensor field and a $\binom{1}{1}$-tensor field. Then we define the functions $\nabla_k \phi_{ij}$, $\nabla_k \phi^{ij}$ and $\nabla_k \phi_i^j$ by

$$\nabla_k \phi_{ij} = \partial_k \phi_{ij} - \left\{ \begin{matrix} & l & \\ k & & i \end{matrix} \right\} \phi_{lj} - \left\{ \begin{matrix} & l & \\ k & & j \end{matrix} \right\} \phi_{il}, \tag{4.27}$$

$$\nabla_k \phi^{ij} = \partial_k \phi^{ij} + \left\{ \begin{matrix} & i & \\ k & & l \end{matrix} \right\} \phi^{lj} + \left\{ \begin{matrix} & j & \\ k & & l \end{matrix} \right\} \phi^{il}, \tag{4.28}$$

$$\nabla_k \phi_i^j = \partial_k \phi_i^j + \left\{ \begin{matrix} & j & \\ k & & l \end{matrix} \right\} \phi_i^l - \left\{ \begin{matrix} & l & \\ k & & i \end{matrix} \right\} \phi_l^j. \tag{4.29}$$

□

Lemma 4.2.4 The components of the first fundamental vector field behave by covariant differentiation like constants, or $\nabla_k g^{ij} = 0$ and $\nabla_k g_{ij} = 0$.

Proof Out of Formula 4.22 follows

$$\partial_k g^{ij} = \partial_k \left(\mathrm{d}u^i, \mathrm{d}u^j\right) = \left(\mathrm{d}u^i, \frac{\nabla}{\mathrm{d}u^k}\right) + \left(\frac{\nabla}{\mathrm{d}u^k}\mathrm{d}u^i, \mathrm{d}u^j\right),$$

such that

$$\partial_k g^{ij} = -\left\{ {j \atop l \quad k} \right\} g^{il} - \left\{ {i \atop l \quad k} \right\} g^{lj},$$

where is made use of Formula 4.23. With the help of Definition 4.28 follows then $\nabla_k g^{ij} = 0$. In a similar way it is to see that $\nabla_k g_{ij} = 0$. □

Definition 4.2.10 Let $\mathbf{v}$ be a tangent vector field on K at S and write $\mathbf{v} = u^i\,\partial_i X$. This tangent vector field is called parallel (transported) along K if

$$\frac{\nabla}{\mathrm{d}t} v^i \partial_i X = \left(\frac{\mathrm{d}v^j}{\mathrm{d}t} + \left\{ {j \atop i \quad k} \right\} \frac{\mathrm{d}u^i}{\mathrm{d}t} v^k\right) \partial_j X = 0. \tag{4.30}$$

□

Note that a curve K is a geodesic if and only if the tangent vector field of this curve is parallel transported along this curve. If K is a geodesic than there holds that

$$\frac{\nabla}{\mathrm{d}t}\left(\frac{\mathrm{d}u^i}{\mathrm{d}t}\partial_i X\right) = 0.$$

<u>IMPORTANT NOTE</u>
The system of differential equations for parallel transport in 2 dimensions reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} + \begin{pmatrix} \left\{ {1 \atop l \quad 1} \right\} \frac{\mathrm{d}u^l}{\mathrm{d}t} & \left\{ {1 \atop l \quad 2} \right\} \frac{\mathrm{d}u^l}{\mathrm{d}t} \\ \left\{ {2 \atop l \quad 1} \right\} \frac{\mathrm{d}u^l}{\mathrm{d}t} & \left\{ {2 \atop l \quad 2} \right\} \frac{\mathrm{d}u^l}{\mathrm{d}t} \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is mostly a nonautonomous coupled system of ordinary linear differential equations of the form

$$\begin{cases} \dfrac{\mathrm{d}v^1}{\mathrm{d}t} + A_{11}(t)\, v^1 + A_{12}(t)\, v^2 = 0, \\ \dfrac{\mathrm{d}v^2}{\mathrm{d}t} + A_{21}(t)\, v^1 + A_{22}(t)\, v^2 = 0. \end{cases}$$

Section 4.3 Exercises

1. Let in $\mathbb{R}^3$, S be the spherical surface, parametrized by

$$\underline{x}(u, v) = (\sin u \cos v, \sin u \sin v, \cos u) \text{ with } 0 < u < \pi,\ 0 \leq v \leq 2\pi.$$

a. Let $T_{\underline{x}}(S)$ be tangent plane to S in some arbitrary point $\underline{x}$ of S. Determine the basis $\partial_u\underline{x}$, $\partial_v\underline{x}$, which belongs to the coordinates u, v, of $T_{\underline{x}}(S)$.
b. Determine the fundamental tensor field g_{ij} of S.
c. Determine a normal vector field $\underline{N}$ on S.
d. Determine the second fundamental tensor field h_{ij} of S.
e. Calculate the dual (= reciproke) basis du, du to $\partial_u\underline{x}$, $\partial_v\underline{x}$.
(Remark: With g_{ij} is $T_{\underline{x}}(S)$ identified with its dual $T^*_{\underline{x}}(S)$.)
f. Determine the Christoffel symbols of S.
g. What are the equation of the geodesics on S?
Argue that the parametrised curves "v =constant" are geodesics.
h. Let de curve K be given by

$$u = \frac{\pi}{4},\ v = t,\ 0 \leq t \leq 2\pi.$$

This curve starts and ends in the point

$$\underline{a} = \left(\frac{1}{2}\sqrt{2}, 0, \frac{1}{2}\sqrt{2}\right).$$

Transport the vector (0, 1, 0) out of $T_{\underline{a}}(S)$ parallel along the curve K.
Is K a geodisc?

2. Let in $\mathbb{R}^3$, S be the pseudosphere, parametrized by

$$\underline{x}(u, v) = (\sin u \cos v, \sin u \sin v, \cos u + \log\tan\frac{u}{2}) \text{ with } 0 < u < \pi,\ 0 \leq v \leq 2\pi.$$

a. Let $T_{\underline{x}}(S)$ be tangent plane to S in some arbitrary point $\underline{x}$ of S. Determine the basis $\partial_u\underline{x}$, $\partial_v\underline{x}$, which belongs to the coordinates u, v, of $T_{\underline{x}}(S)$.
b. Determine the fundamental tensor field g_{ij} of S.
c. Determine a normal vector field $\underline{N}$ on S.
d. Determine the second fundamental tensor field h_{ij} of S.
e. Calculate the dual (= reciproke) basis du, du to $\partial_u\underline{x}$, $\partial_v\underline{x}$.
(Remark: With g_{ij} is $T_{\underline{x}}(S)$ identified with its dual $T^*_{\underline{x}}(S)$.)
f. Determine the Christoffel symbols of S.
Remark: $\left\{ {1 \atop 1\ \ 2} \right\} = \left\{ {2 \atop 1\ \ 1} \right\} = \left\{ {2 \atop 2\ \ 2} \right\} = 0.$
g. What are the equation of the geodesics on S?
Argue that the parametrised curves "v =constant" are geodesics.
h. Let de curve K be given by

$$u = \frac{\pi}{4},\ v = t,\ 0 \leq t \leq 2\pi.$$

This curve starts and ends in the point

$$\underline{a} = \left(\frac{1}{2}\sqrt{2}, 0, \frac{1}{2}\sqrt{2} + \log\left(\sqrt{2} - 1\right)\right).$$

Transport the vector (0, 1, 0) out of $T_{\underline{a}}(S)$ parallel along the curve K. Is K a geodisc?

3.

Section 4.4 RRvH: Christoffel symbols?

4.4.1 Christoffel symbols

This section is just written to get some feeling about what the Christoffel symbols symbolise. Let X be some point in space, with the curvilinear coordinates x^i. The coordinates depend on the variables ξ^j, so $x^i(\xi^j)$. The vector $\partial_j X$ is tangent to the coordinate curve of x^j and the vectors $\{\partial_i X\}$ form a basis. To this basis belongs the reciprocal basis $\{y^j\}$, that means that $(y^j, \partial_i X) = \delta^j_i$. If the standard inner product is used, they can be calculated by taking the inverse of the matrix $(\partial_1 X \cdots \partial_N X)$.
If the point X is moved, not only the coordinates x^i change, but also the basis vectors $\{\partial_i X\}$ and the vectors of the reciprocal basis $\{y^j\}$.
The vector $\partial_j(\partial_i X)$ can be calculated and expanded in terms of the basis $\{\partial_i X\}$, the coeeficients of this expansion are the Christoffel symbols, so

$$\partial_j \partial_i X = \left\{ \begin{matrix} & k & \\ j & & i \end{matrix} \right\} \partial_k X,$$

see also the similarity with Definition 3.6.4. Another notation for the Christoffel symbols is Γ^k_{ji}. The symbols $\left\{ \begin{matrix} & k & \\ j & & i \end{matrix} \right\}$ and Γ^k_{ji} are often called Christoffel symbols of the second kind.

> **Comment(s): 4.4.1** Important to note is that the Christoffel symbol is NOT a tensor with respect to an arbitrary coordinate system.

Using the reciprocal basis, the Christoffel symbols can be calculated by

$$(y^n, \partial_j \partial_i X) = \left\{ \begin{matrix} & n & \\ j & & i \end{matrix} \right\}.$$

Most of the time the metric tensor is used to calculate the Christoffel symbols. The metric tensor is

$$G = (\partial_1 X \cdots \partial_N X)^T (\partial_1 X \cdots \partial_N X),$$

the matrix with all the inner products between the tangent vectors to the coordinate axis. The inverse matrix of G is also needed and the derivatives of all these inner products. The inner products between different tangent vectors is notated by the coefficient $g_{ij} = (\partial_i X, \partial_j X)$ of the matrix G.

And with all of these, there can be constructed the formula

$$\Gamma^k_{lm} = \frac{1}{2} g^{kn} \left(g_{mn,l} + g_{nl,m} - g_{lm,n} \right),$$

where g^{kn} are coefficients out of the matrix G^{-1} and $g_{mn,l} = \frac{\partial g_{mn}}{\partial x^l}$. The expression $\frac{1}{2} \left(g_{mn,l} + g_{nl,m} - g_{lm,n} \right)$ is called a Christoffel symbol of the first kind and is often notated by $[ln, m] = \gamma_{lnm} = g_{rm} \begin{Bmatrix} & r & \\ l & & n \end{Bmatrix}$.

For a little program to calculate the Christoffel symbols,
see (van Hassel, 2010) ,Program to calculate Christoffel symbols.

Chapter 5 Manifolds

Section 5.1 Differentiable Functions

Let U and V be open subsets of respectively $\mathbb{R}^n$ and $\mathbb{R}^m$. Let f be a function from U to V and $\mathbf{a} \in U$.

Definition 5.1.1 The function f is called differentiable in $\mathbf{a}$, if there exists a linear transformation $\mathcal{A}$ of $\mathbb{R}^n$ to $\mathbb{R}^m$ such that for all $\mathbf{h}$ in a small enough neighbourhood of $\mathbf{a}$ holds that

$$f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + \mathcal{A}\mathbf{h} + \mid \mathbf{h} \mid \mathbf{r}(\mathbf{h}),$$

with

$$\lim_{|\mathbf{h}| \to 0} \mid \mathbf{r}(\mathbf{h}) \mid = 0.$$

$\square$

Take cartesian coordinates at $\mathbb{R}^n$ and $\mathbb{R}^m$. Let f^k be the k-th component function of $\mathbf{f}$ and write $\mathbf{a} = \left[a^k\right]$. Let $A = \left[A_i^j\right]$ be the matrix of $\mathcal{A}$ with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. Look in particular the component function f^i and $\mathbf{h} = h\delta_i^j E_j$, than holds

$$f^i(\mathbf{a} + \mathbf{h}) = f^i(a^1, \cdots, a^{j-1}, a^j + h, \cdots, a^n) = f^i(\mathbf{a}) + A_j^i h + o(h), h \to 0.$$

It follows that

$$A_j^i = \frac{\partial f^i}{\partial x^j}(\mathbf{a}).$$

The linear transformation $\mathcal{A}$ is called the derivative in $\mathbf{a}$ and the matrix A is called the functional matrix. For $\mathcal{A}$, we use also the notation $\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}X}(\mathbf{a})$. If $m = n$, than can also be determined the determinant of A. This determinant is just $\frac{\partial f^1, \cdots, f^n}{\partial x^1, \cdots, x^n}(\mathbf{a})$, the Jacobi determinant of f in $\mathbf{a}$.
Let K be a curve in U with parameter $t \in (-\alpha, \alpha)$, for a certain value of $\alpha > 0$. So $K : t \to X(t)$. Let $\mathbf{a} = X(0)$. The tangent vector at K in the point $\mathbf{a}$ is given by $\frac{\mathrm{d}X}{\mathrm{d}t}(0)$. Let L be the image curve in V of K under f. So $L : t \to Y(t) = f(X(t))$. Call $\mathbf{b} = Y(0) = f(\mathbf{a})$. The tangent vector at L in the point $\mathbf{b}$ is given by $\frac{\mathrm{d}Y}{\mathrm{d}t}(0) = \frac{\mathrm{d}f}{\mathrm{d}X}(\mathbf{a}) \frac{\mathrm{d}X}{\mathrm{d}t}(0)$.
If two curves K_1 and K_2 through $\mathbf{a}$ at $\mathbf{a}$ have an identical tangent vector than it follows that the image curves L_1 and L_2 of respectively K_1 and K_2 under f have also an identical

tangent vector.
The three curves K_1, K_2 and K_3 through $\mathbf{a}$ have at $\mathbf{a}$ tangent vectors, which form an addition parallelogram. There holds than that the tangent vectors at the image curves L_1, L_2 and L_3 also form an addition parallelogram.

Section 5.2 Manifolds

Let M be a set[2 3].

Definition 5.2.1 A subset U of M is called a chart ball of M if there exists an open subset $\tilde{U}$ of $\mathbb{R}^n$ such that there exists a map ϕ which maps U bijective on $\tilde{U}$. The open subset $\tilde{U}$ of $\mathbb{R}^n$ is called chart and the map ϕ is called chart map. □

Definition 5.2.2 Let $\phi : U \to \tilde{U}$ and $\psi : V \to \tilde{V}$ be chart maps of M, with $U \cap V \neq \emptyset$. The maps $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are called transition maps. □

Note that transition maps only concern points which occur in more than one chart and they map open subsets in $\mathbb{R}^n$ into open subsets of $\mathbb{R}^n$.

Definition 5.2.3 A collection of chart balls and there corresponding chart maps $\{U_i, \phi_i\}$ of M is called an atlas of M if $M = \cup_i U_i$ and if every transition map is differentiable in the points where they are defined. □

Definition 5.2.4 The set M is called a manifold of dimension n if M is provided with an atlas of which all charts are subsets of $\mathbb{R}^n$. □

Strictly spoken the following has to be added to Definition 5.2.4. M is a topological Hausdorff space which is locally homeomorf with $\mathbb{R}^n$.

In the remainder there is supposed that M is a manifold. Let U and U' be chart balls of M such that $U \cap U' \neq \emptyset$. Let also $\tilde{U}$ and $\tilde{U}'$ be the corresponding charts, with the

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[2] RRvH: Problem is how to describe that set, for instance, with the help of Euclidean coordinates?

[3] RRvH: It is difficult to translate Dutch words coined by the author. So I have searched for English words, commonly used in English texts, with almost the same meaning. The book of (Ivancevic and Invancevic, 2007) ,Applied Differential Geometry was very helpful.

chart maps $\phi : U \to \tilde{U}$ and $\phi' : U' \to \tilde{U}'$. Let $\tilde{U}$ and $\tilde{U}'$ be provided with the respective coordinates u^i and $u^{i'}$ and write for $x \in U \cap U'$,

$$\phi(x) = (u^1, \cdots, u^n) \text{ and } \phi'(x) = (u^{1'}, \cdots, u^{n'}).$$

There holds that

$$\left(\phi \circ (\phi')^{-1}\right)(u^{1'}, \cdots, u^{n'}) = (u^1, \cdots, u^n) \text{ and } \left((\phi') \circ \phi^{-1}\right)(u^1, \cdots, u^n) = (u^{1'}, \cdots, u^{n'}),$$

which we simply write as $u^i = u^i(u^{i'})$ and $u^{i'} = u^{i'}(u^i)$. Because of the fact that the transition maps[4] are differentiable, we can introduce transition matrices by

$$A^{i'}_i = \frac{\partial u^{i'}}{\partial u^i} \text{ and } A^i_{i'} = \frac{\partial u^i}{\partial u^{i'}}.$$

Chart maps are bijective and there holds that $\det[A^{i'}_i] \neq 0$ and $\det[A^i_{i'}] \neq 0$. The intersection $U \cap U'$ of M is apparently parametrised by (at least) two curvilinear coordinate systems.

In the remainder U and U' are chart balls of M with a non-empty intersection $U \cap U'$. The corresponding charts and chart maps we notate respectively with $\tilde{U}$, $\tilde{U}'$ and ϕ, ϕ'. The open subsets $\tilde{U}$, $\tilde{U}'$ are provided by the coordinates u^i and $u^{i'}$.

Definition 5.2.5 A curve K at M is a continuous injective map of an open interval I to M. □

Let K be curve at M such that a part of the curve lies at $U \cap U'$. That part is a curve that appears at the chart $\tilde{U}$, at the chart $\tilde{U}'$ and is a curve in $\mathbb{R}^n$. A point $X(t_0) \in U \cap U'$, for a certain $t_0 \in I$, can be found at both charts $\tilde{U}$ and $\tilde{U}'$. At these charts the tangent vectors at K in $X(t_0)$ are given by

$$\frac{\mathrm{d}(\phi \circ X)}{\mathrm{d}t}(t_0) \text{ and } \frac{\mathrm{d}((\phi') \circ X)}{\mathrm{d}t}(t_0).$$

Let $K_1 : t \mapsto X(t), t \in I_1$ and $K_2 : \tau \mapsto Y(\tau), \tau \in I_2$ be curves at M, which have a point P in common in $U \cap U'$, say $P = X(t_0) = Y(\tau_0)$, for certain $t_0 \in I_1$ and $\tau_0 \in I_2$. Suppose that the tangent vectors on K_1 and K_2 in P at the chart $\tilde{U}$ coincide. The tangent vectors on K_1 and K_2 in P at the chart $\tilde{U}'$ also coincide, because by changing of chart these tangent vectors transform with the transition matrix $A^{i'}_i$.

Definition 5.2.6 Two curves K_1 and K_2 at M which both have the point P in common are called equivalent in P, if the tangent vectors on K_1 and K_2 in P at a chart $\tilde{U}$ coincide. From the above it follows that this definition is chart independent. □

[4] RRvH: Transition maps represent in general a nonlinear transformation of coordinates.

Definition 5.2.7 Let $P \in M$. A class of equivalent curves in P is called a tangent vector in P at M. The set of all tangent vectors in P at M is called the tangent space in P. □

Note that the tangent space is a vector space of dimension n. We notate these tangent vectors by their description at the charts. A basis of the tangent space is formed by the tangent vectors $\dfrac{\partial}{\partial u^i}$ in P at the parameter curves, which belong to the chart $\tilde{U}$. The relationship of the tangent vectors $\dfrac{\partial}{\partial u^{i'}}$, which belong to the parameter curves of chart $\tilde{U}'$, is given by

$$\frac{\partial}{\partial u^{i'}} = A^i_{i'} \frac{\partial}{\partial u^i}.$$

Definition 5.2.8 A function at M is a map of a part of M to the real numbers. □

At the chart $\tilde{U}$ is a function f described by $f\,\phi^{-1} : \tilde{U} \to \mathbb{R}$. Note that $\tilde{U} = \phi(U)$. We notate also functions by there description at the charts.

Definition 5.2.9 Two functions f and g at M are called equivalent in a point P if for their descriptions $f(u^i)$ and $g(u^i)$ at U holds

$$\partial_j f(u^i_0) = \partial_j g(u^i_0),$$

whereby $\phi(P) = (u^1_0, \cdots, u^n_0)$. □

The foregoing definition is chart independent because

$$u^{i'} = u^{i'}(u^i),\ \partial_{j'} f = A^j_{j'}\, \partial_j f.$$

Definition 5.2.10 A covector in P at the manifold M is a class of in P equivalent functions. The cotangent space in P is the set of the covectors in P at M. □

The cotangent space is a vector space of dimension n. The covectors $\mathrm{d}u^i$ in P of the parameter functions u^i, which belong to the chart $\tilde{U}$, form a basis of the cotangent space. For two charts holds

$$\mathrm{d}u^{i'} = A^{i'}_i\, \mathrm{d}u^i.$$

For a function f and a curve K in a point P of M is $f \circ K$ a map of the open interval I to $\mathbb{R}$ and there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ K) = \frac{\mathrm{d}}{\mathrm{d}t} f(u^i(t)) = \partial_i f \frac{\mathrm{d}u^i}{\mathrm{d}t}.$$

This expression, which is chart independent, is called the directional derivative in P of the function f with respect to the curve K and is conform the definition 3.6.2, Subsection 3.6.1. In the directional derivative in P we recognize the covectors as linear functions at the tangent space and the tangent vectors as linear functions at the cotangent space. The tangent space and the cotangent space in P can therefore be considered as each other's dual.
The tangent vectors in a point P at the manifold M, we can also define as follows:

Definition 5.2.11 The tangent vector in P is a linear transformation $\mathcal{D}$ of the set of functions at M, which are defined in P, in $\mathbb{R}$, which satisfies

$$\mathcal{D}(\alpha f + \beta g) = \alpha \mathcal{D}(f) + \beta \mathcal{D}(g), \mathcal{D}(f g) = f \mathcal{D}(g) + g \mathcal{D}(f). \tag{5.1}$$

□

A tangent vector according definition 5.2.7 can be seen as such a linear transformation. Let K be a curve and define

$$\mathcal{D} f = \frac{\mathrm{d}}{\mathrm{d}t} f \circ K,$$

than $\mathcal{D}$ satisfies 5.1, because for constants α and β holds

$$\frac{\mathrm{d}u^i}{\mathrm{d}t}\partial_i(\alpha f + \beta g) = \alpha \frac{\mathrm{d}u^i}{\mathrm{d}t}\partial_i f + \beta \frac{\mathrm{d}u^i}{\mathrm{d}t}\partial_i g, \ \frac{\mathrm{d}u^i}{\mathrm{d}t}\partial_i(f g) = f \frac{\mathrm{d}u^i}{\mathrm{d}t}\partial_i g + g \frac{\mathrm{d}u^i}{\mathrm{d}t}\partial_i f.$$

Section 5.3 Riemannian manifolds

Let M be a manifold.

Definition 5.3.1 A tensor field at M is a map which adds to every point of M a tensor of the corresponding tangent space of that point. □

In every tangent space can be introduced an inner product with which tensoralgebra can be done. Unlike surfaces in $\mathbb{R}^3$ we don't have in general an a priori given inner product, that can be used simultaneously to all the tangent spaces. This missing link between the different tangent spaces of a manifold is completed in the following definition.

Definition 5.3.2 A Riemannian manifold is a manifold which is equipped with a smooth, symmetric and positive definite 2-tensor field. □

Let now be M a Riemannian manifold. To every point $P \in M$ and every pair $\mathbf{v}$, $\mathbf{w} \in T_P(M)$ belongs a function $\gamma_P : (\mathbf{v}, \mathbf{w}) \mapsto \gamma(P; \mathbf{v}, \mathbf{w})$ such that γ_P is linear in $\mathbf{v}$ and $\mathbf{w}$, is symmetric in $\mathbf{v}$ and $\mathbf{w}$ and also satisfies to $\gamma(P; \mathbf{v}, \mathbf{v}) > 0$ if $\mathbf{v} \neq \mathbf{0}$.
Let U be a chart ball of M with the corresponding chart map ϕ and coordinates $\{u^i\}$. Let v^i and w^i be the components of respectively $\mathbf{v}$ and $\mathbf{w}$ with respect to these u^i and write

$$\gamma(P; \mathbf{v}, \mathbf{w}) = g_{ij} \, v^i \, w^j,$$

with $g_{ij} = g_{ji}$ and $[g_{ij}]$ positive definite. In the tangent space $T_P(M)$ serves γ as fundamental tensor. Therefore we call the 2-tensor field that belongs to M the fundamental tensor field.

To a given fundamental tensor field can be introduced Christoffel symbols by

$$\left\{ \begin{matrix} & i & \\ j & & k \end{matrix} \right\} = \frac{1}{2} \, g^{il} \, (\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}).$$

We call the curves which satisfy the differential equations

$$\ddot{u}^k + \left\{ \begin{matrix} & k & \\ i & & j \end{matrix} \right\} \dot{u}^i \, \dot{u}^j = 0$$

geodesics of the Riemannian manifold. Even as in the chapter before can be proved that the shortest curves in a Riemannian manifold are geodesics.

Example(s): 5.3.1 a mechanical system with n degrees of freedom, with generalised coordinates $q^1, \cdots, q^n$, of which the kinetic energy is a positive definite quadratic norm in $\dot{q}^i$, with coefficients which depend on q^i,

$$T = \frac{1}{2} a_{ij}(q^k)\, \dot{q}^i\, \dot{q}^j.$$

The differential equations of the behaviour of the system are the equations of Lagrange,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}^k}\right) - \frac{\partial T}{\partial q^k} = K_k,$$

where $K_k(q^j, t)$ are the generalised outside forces.
The configuration of the system forms a Riemannian manifold of dimension n, T appears as fundamental tensor. The equations of Lagrange can be written as

$$\ddot{q}k + \left\{ \begin{matrix} & k & \\ i & & j \end{matrix} \right\} \dot{q}^i\, \dot{q}^j = K^k.$$

When there work no outside forces at the system a geodesic orbit is followed by the system on the Riemannian manifold.

Notice(s): 5.3.1 According the definition of a Riemannian manifold is every tangent space provided with a positive definite fundamental tensor. With some effort and some corrections, the results of this paragraph and the paragraphs that follow can be made valid for Riemannian manifolds with an indefinite fundamental tensor. Herein is every tangent space a Minkowski space. This note is made in relation to the furtheron given sketch about the general theory of relativity.

Section 5.4 Covariant derivatives

Regard a Riemannian manifold M formed by an open subset of $\mathbb{R}^n$ and a curvilinear coordinate system $\{u^i\}$ at M as a chart. Let $\mathbf{a}$ be a constant vector field on M. While $\mathbf{a}$ is constant with respect to the coordinates u^i of the chart, the components a^i of $\mathbf{a}$, with respect to the basis $\partial_i X$ of the tangent space, depend on u^i, because the basis $\partial_i X$ depends on u^i. We differentiate this constant vector field $\mathbf{a}$ along a curve K, described by $X(t) = X(u^i(t))$. Write $\mathbf{a} = v^i(t)\partial_i X(t)$ at every point of K. Then holds

$$\mathbf{0} = \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(v^i\, \partial_i X) = \left(\frac{\mathrm{d}v^i}{\mathrm{d}t} + \left\{ \begin{matrix} & i & \\ j & & k \end{matrix} \right\} \frac{\mathrm{d}u^j}{\mathrm{d}t} v^k\right) \partial_i X.$$

This gives us the idea to define the covariant derivative of a vector field $\mathbf{w} = w^i\partial_i X$ along a curve K by

$$\left(\frac{\nabla}{\mathrm{d}t}w^i\right)\partial_i X = \left(\frac{\mathrm{d}w^i}{\mathrm{d}t} + \left\{ {i \atop j\quad k} \right\}\frac{\mathrm{d}u^j}{\mathrm{d}t}w^k\right)\partial_i X. \tag{5.2}$$

Here is a vector field along a curve K. This vector field is independent of the choice of the coordinates,

$$\frac{\mathrm{d}w^{i'}}{\mathrm{d}t} + \left\{ {i' \atop j'\quad k'} \right\}\frac{\mathrm{d}u^{j'}}{\mathrm{d}t}w^{k'} =$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(A^{i'}_i w^i\right) + \left(A^{i'}_i A^j_{j'} A^k_{k'}\left\{ {i \atop j\quad k} \right\} + A^{i'}_s \partial_{j'}\left(A^s_{k'}\right)\right)A^{j'}_p \frac{\mathrm{d}u^p}{\mathrm{d}t} A^{k'}_q w^q =$$

$$A^{i'}_i\left(\frac{\mathrm{d}w^i}{\mathrm{d}t} + \left\{ {i \atop j\quad k} \right\}\frac{\mathrm{d}u^j}{\mathrm{d}t}w^k\right) + \partial_h\left(A^{i'}_i\right)\frac{\mathrm{d}u^h}{\mathrm{d}t}w^i + A^{i'}_s\,\partial_p\left(A^s_{k'}\right)A^{k'}_q\frac{\mathrm{d}u^p}{\mathrm{d}t}w^q.$$

The last term in the expression above is equal to

$$-A^{i'}_s A^s_{k'}\,\partial_p\left(A^{k'}_q\right)\frac{\mathrm{d}u^p}{\mathrm{d}t}w^p = -\partial_p\left(A^{i'}_q\right)\frac{\mathrm{d}u^p}{\mathrm{d}t}w^k,$$

such that

$$\frac{\mathrm{d}w^{i'}}{\mathrm{d}t} + \left\{ {i' \atop j'\quad k'} \right\}\frac{\mathrm{d}u^{j'}}{\mathrm{d}t}w^{k'} = A^{i'}_i\left(\frac{\mathrm{d}w^i}{\mathrm{d}t} + \left\{ {i \atop j\quad k} \right\}\frac{\mathrm{d}u^j}{\mathrm{d}t}w^k\right).$$

Let M be Riemannian manifold and $\{U, \phi\}$ a chart[5], with coordinates u^i and Christoffel symbols $\left\{ {k \atop l\quad m} \right\}$. Let K be a parametrised curve at the chart U and T a $\binom{r}{s}$ tensor field, that at least is defined in every point of K.We want to introduce a differentiation operator $\frac{\nabla}{\mathrm{d}t}$ along K such that $\frac{\nabla}{\mathrm{d}t}T$ is an on K defined $\binom{r}{s}$ tensor field. $\frac{\nabla}{\mathrm{d}t}$ is called the covariant derivative along K.

We consider first the case $r = 1$, $s = 0$. The covariant derivative of a tangent vector field of M along K we define by Expression 5.2. If $\frac{\nabla}{\mathrm{d}t}\mathbf{a} = \mathbf{0}$ delivers, we call $\mathbf{a}$ pseudoparallel along the curve K. Out of the theory of the ordinary differential equations follows that a tangent vector $\mathbf{a_0}$ on M given at the begin of the curve K can be continued to a pseudoparallel vector field along K. In other words, $\mathbf{a_0}$ can be parallel transported along the curve K.

Notice that the geodesics are exactly those curves, where with the use of the arclength parametrisation, the tangent vectors are pseudoparallel to the curve. There holds

[5] RRvH: The definition of a chart ball and a chart is not consequently used by the author. $\tilde{U} = \phi(U)$ is a chart of M, $\{U, \phi\}$ is a chart ball of M. The cause of this confusion is the use of the lecture notes of Prof. Dr. J.J. Seidel, see (Seidel, 1980) ??.

$$\frac{\nabla \dot{u}^k}{\mathrm{d}s} = \frac{\mathrm{d}\dot{u}^k}{\mathrm{d}s} + \left\{ \begin{matrix} & k & \\ i & & j \end{matrix} \right\} \dot{u}^i \, \dot{u}^j = 0.$$

We consider now the case $r = 0, s = 1$, the covariant derivative of a covector field (of the covariant components of a vector field) along the curve K. Let $\theta_r \, \mathrm{d}u^r$ be a given covector field that at least is defined everywhere on K and $a^r \, \partial_r X$ a pseudoparallel vector field along K. Then holds

$$\frac{\mathrm{d}}{\mathrm{d}t} (a^r \, \theta_r) = \frac{\mathrm{d}a^r}{\mathrm{d}t} \, \theta_r + a^r \, \frac{\mathrm{d}\theta_r}{\mathrm{d}t}$$
$$= -\left\{ \begin{matrix} & r & \\ j & & k \end{matrix} \right\} \frac{\mathrm{d}u^j}{\mathrm{d}t} \, a^k \, \theta_r + a^r \, \frac{\mathrm{d}\theta_r}{\mathrm{d}t} + \theta_r \, \frac{\nabla}{\mathrm{d}t} a^r$$
$$= \left(\frac{\mathrm{d}\theta_k}{\mathrm{d}t} - \left\{ \begin{matrix} & r & \\ j & & k \end{matrix} \right\} \frac{\mathrm{d}u^j}{\mathrm{d}t} \, \theta_r \right) a^k.$$

If we want that the Leibniz rule holds, by taking the covariant derivative, then we have to define

$$\frac{\nabla}{\mathrm{d}t} \theta_k = \frac{\mathrm{d}\theta_k}{\mathrm{d}t} - \left\{ \begin{matrix} & r & \\ j & & k \end{matrix} \right\} \frac{\mathrm{d}u^j}{\mathrm{d}t} \, \theta_r, \tag{5.3}$$

along K.
There can be directly proved that by the change of a chart holds

$$\frac{\nabla}{\mathrm{d}t} \theta_{k'} = A^k_{k'} \, \frac{\nabla}{\mathrm{d}t} \theta_k.$$

Analogously an arbitrary 2-tensor field ϕ is treated. Take for instance $r = 0, s = 2$ and notate the components of ϕ by ϕ_{ij}. Take two arbitrary pseudoparallel vector fields $\mathbf{a} = a^i \, \partial_i X$ and $\mathbf{b} = b^j \, \partial_j X$, along K and require that the Leibniz rule holds then

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\phi_{ij} \, a^i \, b^j \right) = \frac{\mathrm{d}a^i}{\mathrm{d}t} \, \phi_{ij} \, b^j + \frac{\mathrm{d}b^j}{\mathrm{d}t} \, \phi_{ij} \, a^i + a^i \, b^j \, \frac{\mathrm{d}\phi}{\mathrm{d}t}$$
$$= \left(\frac{\mathrm{d}\phi_{kl}}{\mathrm{d}t} - \left\{ \begin{matrix} & j & \\ m & & k \end{matrix} \right\} \frac{\mathrm{d}u^m}{\mathrm{d}t} \, \phi_{jl} - \left\{ \begin{matrix} & i & \\ m & & l \end{matrix} \right\} \frac{\mathrm{d}u^m}{\mathrm{d}t} \, \phi_{ki} \right) a^k \, b^l.$$

So we have to define

$$\frac{\nabla}{\mathrm{d}t} \phi_{kl} = \frac{\mathrm{d}\phi_{kl}}{\mathrm{d}t} - \left\{ \begin{matrix} & m & \\ j & & k \end{matrix} \right\} \frac{\mathrm{d}u^j}{\mathrm{d}t} \, \phi_{ml} - \left\{ \begin{matrix} & n & \\ j & & l \end{matrix} \right\} \frac{\mathrm{d}u^j}{\mathrm{d}t} \, \phi_{kn}. \tag{5.4}$$

By change of a chart it turns out again that

$$\frac{\nabla}{\mathrm{d}t} \phi_{k'l'} = A^k_{k'} \, A^l_{l'} \, \frac{\nabla}{\mathrm{d}t} \phi_{kl}.$$

Simili modo the case that $r = 1, s = 1$ is tackled by the contraction of a pseudoparallel vector field and a ditto covector field. This delivers

$$\frac{\nabla}{dt}\phi_l^k = \frac{d}{dt}\phi_l^k + \left\{ \begin{matrix} k \\ j \quad p \end{matrix} \right\} \frac{du^j}{dt}\,\phi_l^p - \left\{ \begin{matrix} r \\ j \quad l \end{matrix} \right\} \frac{du^j}{dt}\,\phi_r^k. \tag{5.5}$$

To higher order tensors it goes the same way. Taking the covariant derivative along a parametrised curve means, take first the 'normal' derivative and then add for every index a Christoffel symbol.

For the curve K we choose a special curve, namely the h-th parameter curve. So $u^j = K^j + \delta^{jh}\,t$, with K^j constants. So there holds $t = u^h - K^h$. With $\frac{\nabla}{dt} = \frac{\nabla}{du^h} = \nabla_h$ we find that

$$\begin{aligned}
\nabla_h w^i &= \partial_h w^i + \left\{ \begin{matrix} i \\ h \quad k \end{matrix} \right\} w^k, \\
\nabla_h \theta_k &= \partial_h \theta_k - \left\{ \begin{matrix} r \\ h \quad k \end{matrix} \right\} \theta_r, \\
\nabla_h \phi_{kl} &= \partial_h \phi_{kl} - \left\{ \begin{matrix} r \\ h \quad k \end{matrix} \right\} \phi_{rl} - \left\{ \begin{matrix} s \\ h \quad l \end{matrix} \right\} \phi_{ks}, \\
\nabla_h \phi_i^{jk} &= \partial_h \phi_i^{jk} - \left\{ \begin{matrix} m \\ h \quad i \end{matrix} \right\} \phi_m^{jk} + \left\{ \begin{matrix} j \\ h \quad m \end{matrix} \right\} \phi_i^{mk} + \left\{ \begin{matrix} k \\ h \quad m \end{matrix} \right\} \phi_i^{jm}, \\
\nabla_h g_{ij} &= 0, \text{ etc., etc..}
\end{aligned}$$

If there is changed of a chart, the next case behaves (check!) as

$$\nabla_{h'} \phi_{i'}^{j'k'} = A_{h'}^h\, A_{i'}^i\, A_j^{j'}\, A_k^{k'}\, \nabla_h \phi_i^{jk}.$$

The covariant derivative along all parameter curves converts a $\binom{r}{s}$-tensor vector field into a $\binom{r}{s+1}$-tensor on M. Most of the time the covariant derivative is interpreted as the latter.

Section 5.5 The curvature tensor

If a function f of two variables x and y is smooth enough, there holds that

$$\frac{\partial^2 f}{\partial x\, \partial y} - \frac{\partial^2 f}{\partial y\, \partial x} = 0.$$

However, the second covariant derivative of a vector field is not symmetric. There holds

$$\nabla_h \nabla_i v^k = \partial_h \left(\nabla_i v^k \right) - \left\{ \begin{matrix} m \\ h \quad i \end{matrix} \right\} \nabla_m v^k + \left\{ \begin{matrix} k \\ h \quad m \end{matrix} \right\} \nabla_i v^m$$
$$= \partial_h \partial_i v^k + \left\{ \begin{matrix} k \\ i \quad j \end{matrix} \right\} \partial_h v^j + v^j \partial_h \left\{ \begin{matrix} k \\ i \quad j \end{matrix} \right\} - \left\{ \begin{matrix} m \\ h \quad i \end{matrix} \right\} \nabla_m v^k + \left\{ \begin{matrix} k \\ h \quad m \end{matrix} \right\} \partial_i v^m + \left\{ \begin{matrix} k \\ h \quad m \end{matrix} \right\} \left\{ \begin{matrix} m \\ i \quad j \end{matrix} \right\} v^j.$$

Reverse the rule of h and i and out of the difference with the latter follows

$$\nabla_h \nabla_i v^k - \nabla_i \nabla_h v^k = \left(\partial_h \left\{ \begin{matrix} k \\ i \quad j \end{matrix} \right\} - \partial_i \left\{ \begin{matrix} k \\ h \quad j \end{matrix} \right\} + \left\{ \begin{matrix} k \\ h \quad m \end{matrix} \right\} \left\{ \begin{matrix} m \\ i \quad j \end{matrix} \right\} - \left\{ \begin{matrix} k \\ i \quad m \end{matrix} \right\} \left\{ \begin{matrix} m \\ h \quad j \end{matrix} \right\} \right) v^j.$$

The left hand side is the difference of two $\binom{1}{2}$-tensor fields and so the result is a $\binom{1}{2}$-tensor field. Because of the fact that v^j are the components of a vector field, the expression between the brackets in the right hand side, are components of a $\binom{1}{3}$-tensor field.

Definition 5.5.1 The curvature tensor of Riemann-Christoffel is a$\binom{1}{3}$-tensor field of which the components are given by

$$K^k_{hij} = \partial_h \left\{ \begin{matrix} k \\ i \quad j \end{matrix} \right\} - \partial_i \left\{ \begin{matrix} k \\ h \quad j \end{matrix} \right\} + \left\{ \begin{matrix} k \\ h \quad m \end{matrix} \right\} \left\{ \begin{matrix} m \\ i \quad j \end{matrix} \right\} - \left\{ \begin{matrix} k \\ i \quad m \end{matrix} \right\} \left\{ \begin{matrix} m \\ h \quad j \end{matrix} \right\}. \tag{5.6}$$

□

The following relations hold

$$\left(\nabla_h \nabla_i - \nabla_i \nabla_h \right) v^k = K^k_{hij} v^j,$$
$$\left(\nabla_h \nabla_i - \nabla_i \nabla_h \right) w_j = K^k_{hij} w_k,$$
$$\left(\nabla_h \nabla_i - \nabla_i \nabla_h \right) \phi^k_j = K^k_{him} \phi^m_j - K^m_{hij} \phi^k_m.$$

On analogous way one can deduce such kind of relations for other type of tensor fields.

With some tenacity the tensorial character of the curvature tensor of Riemann-Christoffel, defined in 5.6, can be verified. This can be done only with the use of the transformation rule:

$$\left\{ \begin{matrix} i' \\ j' \quad k' \end{matrix} \right\} = A^{i'}_i A^j_{j'} A^k_{k'} \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} + A^{i'}_i \partial_{j'} A^i_{k'},$$

see section 3.6.3, formula 3.8.

Notice(s): 5.5.1 Note that:

$$K^k_{hij} = -K^k_{ihj}.$$

In the case that $\left\{ {k \atop i \quad j} \right\} = \left\{ {k \atop j \quad i} \right\}$ then

$$K^k_{hij} + K^k_{jhi} + K^k_{ijh} = 0.$$

In the case, see formula 3.9,

$$\left\{ {k \atop i \quad j} \right\} = \left\{ {k \atop j \quad i} \right\} = g^{km}\,[ij;m] \;\text{ with }\; [ij;m] = \frac{1}{2}\left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right),$$

we find for the covariant components of K

$$\begin{aligned} K_{hijk} &= g_{kl}\,K^l_{hij} \\ &= \partial_h\partial_j g_{ik} + \partial_i\partial_k g_{hj} - \partial_h\partial_k g_{ij} - \partial_i\partial_j g_{hk} + \\ &+ g^{ml}\left([ik;m][hj;l] - [ij;l][hk;m]\right). \end{aligned}$$

It requires a lot of perseverance to verify this formula! We note some symmetries

$$K_{hijk} = K_{ihjk} = K_{hikj} = K_{ihkj} \;\text{ and }\; K_{hijk} + K_{jhik} + K_{ijhk} = 0.$$

Out of this follows that $K_{iijk} = K_{himm} = 0$. In 2 dimensions the curvature tensor seems to be given by just one number. In that case only the following components can be not equal to zero

$$K_{1212} = -K_{1221} = K_{2112} = -K_{2121} = h_{11}\,h_{22} - h_{12}\,h_{21} = \det[g_{ik}]\det[h^k_i].$$

The last identity can be easily proved by choosing just one smart chart. The "tangent plane coordinates" for instance, such as is done in the proof of Theorem 4.2.1.

Through contraction over the components h and k of the Riemann-Christoffeltensor there arises a $\binom{0}{2}$-tensor field of which the components are given by

$$K_{ij} = K^k_{kij} = \partial_k\left\{ {k \atop i \quad j} \right\} - \partial_i\left\{ {k \atop k \quad j} \right\} + \left\{ {k \atop k \quad m} \right\}\left\{ {m \atop i \quad j} \right\} - \left\{ {k \atop i \quad m} \right\}\left\{ {m \atop k \quad j} \right\}.$$

We study this tensor field on symmetry. The first term keeps unchanged if i and j are changed. The combination of the 3th and 4th term also. Only the 2th term needs some further inspection. With

$$\left\{ {k \atop k \quad j} \right\} = g^{km}\,[kj;m] = \frac{1}{2}\,g^{km}\,\partial_j g_{km}$$

we find that

$$\partial_i\left\{ {k \atop k \quad j} \right\} = \frac{1}{2}\left(\partial_i g^{km}\right)\partial_j g_{km} + \frac{1}{2}\,g^{km}\,\partial_i\partial_j g_{km}.$$

The 2th term in the right hand side turns out to be symmetric in i and j. For the 1th term we write with the help of $\partial_i g^{km} = -g^{kr}\,g^{lm}\,\partial_i g_{rl}$ the expression

$$\frac{1}{2}\left(\partial_i g^{km}\right)\partial_j g_{km} = -\,g^{kr}\,g^{lm}\,(\partial_i g_{rl})\left(\partial_j g_{km}\right).$$

Also this one is symmetric in i and j.
Out of this, there can be derived a constant, given by

$$K = K_{hi}\,g^{hi}.$$

With the help of this scalar is formed the Einstein tensor field

$$G_{hi} = K_{hi} - \frac{1}{2}K\,g_{hi}.$$

This tensor field depends only of the components of the fundamental tensor field and plays an important rule in the general relatively theory. It satisfies also the following properties

$$G_{hi} = G_{ih} \;\text{ and }\; \nabla_i G^{hi} = 0.$$

Chapter 6 Appendices

Section 6.1 The General Tensor Concept

On the often heard question: "What is a tensor now *really*?" is a sufficient answer, vague and also sufficient general, the following: "A tensor is a function T of a number of vector variables which are linear in each of these variables separately. Furthermore this function has not to be necessary real-valued, she may take values in another vector space."

Notation(s): Given

- k Vector spaces $E_1, E_2, \cdots, E_k$.
- A vector space F.

Then we note with

$$L^k(E_1, E_2, \cdots, E_k; F)$$

the set of all multilinear functions

$$t : E_1 \times E_2 \times \cdots \times E_k \to F$$
$$(\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k) \mapsto t(\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k) \in F.$$

Comment(s): 6.1.1

a. Multilinear means that for every inlet, for instance the j-th one, holds that

$$t(\underline{u}_1, \cdots, \alpha\,\underline{u}_j + \beta\,\underline{v}_j, \cdots, \underline{u}_k) = \alpha\, t(\underline{u}_1, \cdots, \underline{u}_j, \cdots, \underline{u}_k) + \beta\, t(\underline{u}_1, \cdots, \underline{v}_j, \cdots, \underline{u}_k),$$

for all $\underline{u}_j \in E_j$, for all $\underline{v}_j \in E_j$ and for all $\alpha, \beta \in \mathbb{R}$.

b. The vector spaces E_j, $1 \leq j \leq k$, can all be different and the dimensions can be different.

c. Of $L^k(E_1, E_2, \cdots, E_k; F)$ can be made a vector space by introducing an addition and and a scalar multiplication as follows

$$(\alpha\, t + \beta\,\tau)(\underline{u}_1, \cdots, \underline{u}_k) = \alpha\, t(\underline{u}_1, \cdots, \underline{u}_k) + \beta\,\tau(\underline{u}_1, \cdots, \underline{u}_k).$$

Herein are $t, \tau \in L^k(E_1, E_2, \cdots, E_k; F)$ and $\alpha, \beta \in \mathbb{R}$.

Exercise. If $\dim E_j = n_j$ and $\dim F = m$, calculate then the $\dim L^k(E_1, E_2, \cdots, E_k; F)$.

Notation(s):

- $L^1(E;F) = L(E;F)$ notates the vector space of all linear transformations of E to F.
- $L(E;\mathbb{R}) = E^*$, the dual vector space of E, the vector space of all linear functions on E.
- If $\dim E < \infty$ then $L(E^*;\mathbb{R}) = E^{**} = E$.

Exercise. Let see that $L^k(E_1, \cdots, E_k; F)$, with $\dim F < \infty$, is basically the same as $L^k(E_1, \cdots, E_k, F^*; \mathbb{R})$.

Theorem 6.1.1 There is a natural isomorphism

$$L(E_k, L^{k-1}(E_1, \cdots, E_{k-1}; F)) \simeq L^k(E_1, \cdots, E_k; F)$$

□

Proof Take $\phi \in L(E_k, L^{k-1}(E_1, \cdots, E_{k-1}; F))$ and define $\widetilde{\phi} \in L^k(E_1, \cdots, E_k; F)$ by $\widetilde{\phi} = (\phi(u_k))(\underline{u}_1, \cdots, \underline{u}_{k-1})$. The addition $\phi \mapsto \widetilde{\phi}$ is a isomorphism, i.e. a bijective map. (You put in $\widetilde{\phi}$ a "fixed" vector $\underline{u}_k \in E_k$ at the k-th position and you hold on a multilinear function with $(k-1)$ inlets.) □

Notation(s): If we take for $E_1, \cdots, E_k$ respectively r copies of E^* and s copies of E, with $r + s = k$, and also suppose that $F = \mathbb{R}$, then we write $T^r_s(E)$ in stead of $L^{r+s}(\overbrace{E^*, \cdots, E^*}^{\text{r pieces}}, \overbrace{E, \cdots, E}^{\text{s pieces}}; \mathbb{R})$. The elements of this vector space $T^r_s(E)$ are called (mixed) $(r+s)$-tensors on E, they are called contravariant of the order r and covariant of the order s.

Definition 6.1.1 (Tensorproduct)
Given: $t_1 \in T^{r_1}_{s_1}(E)$, $t_2 \in T^{r_2}_{s_2}(E)$.
Then is the tensorproduct $t_1 \otimes t_2 \in T^{r_1+r_2}_{s_1+s_2}(E)$ defined by

$$(t_1 \otimes t_2)(\underline{\hat{p}}_1, \cdots, \underline{\hat{p}}_{r_1}, \underline{\hat{q}}_1, \cdots, \underline{\hat{q}}_{r_2}, \underline{x}_1, \cdots, \underline{x}_{s_1}, \underline{y}_1, \cdots, \underline{y}_{s_2}) =$$

$$t_1(\underline{\hat{p}}_1, \cdots, \underline{\hat{p}}_{r_1}, \underline{x}_1, \cdots, \underline{x}_{s_1}) \cdot t_2(\underline{\hat{q}}_1, \cdots, \underline{\hat{q}}_{r_2}, \underline{y}_1, \cdots, \underline{y}_{s_2}),$$

with $\underline{\hat{p}}_j, \underline{\hat{q}}_j \in E^*$ and $\underline{x}_j, \underline{y}_j \in E$ arbitrary chosen. □

Comment(s): 6.1.2

a. The product-operation $\otimes$ is not-commutative, associative and bilinear.
b. Because of the previously mentioned identifications we have

$$T^1_0(E) \simeq ET^0_1(E) \simeq E^*,$$
$$T^2_0(E) \simeq L(E; E^*)T^1_1(E) \simeq L(E; E).$$

Theorem 6.1.2 If $\dim(E) = n$ then has $T^r_s(E)$ the structure of a n^{r+s}-dimensional real vector space. The system

$$\{\underline{e}_{i_1} \otimes \underline{e}_{i_r} \otimes \overset{\wedge}{\underline{e}}^{j_1} \otimes \overset{\wedge}{\underline{q}}^{j_s} \mid 1 \leq i_k \leq n, 1 \leq i_k \leq n\},$$

associated with a basis $\{\underline{e}_j\}$ E, forms a basis of $T^r_s(E)$. □

Proof We must show that the previously mentioned system is linear independent in $T^r_s(E)$ and also spans $T^r_s(E)$.
Suppose that

$$\alpha^{i_1 \cdots i_r}_{j_1 \cdots j_s} \underline{e}_{i_1} \otimes \underline{e}_{i_r} \otimes \overset{\wedge}{\underline{e}}^{j_1} \otimes \overset{\wedge}{\underline{q}}^{j_s} = 0$$

fill all systems $(\overset{\wedge}{\underline{e}}^{k_1} \otimes \overset{\wedge}{\underline{e}}^{k_r} \otimes \underline{e}_{l_1} \otimes \underline{e}_{l_s})$, with $1 \leq k_j \leq n$, $1 \leq l_s \leq n$, in the mentioned $(r+s)$-tensor then follows with $< \overset{\wedge}{\underline{e}}^p, \overset{\wedge}{\underline{e}}_q >= \delta^p_q$ that all numbers $\alpha^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ have to be equal to zero. Finally, what concerns the span, every tensor $t \in T^r_s(E)$ can be written as

$$t = t(\overset{\wedge}{\underline{e}}^{i_1} \cdots \overset{\wedge}{\underline{e}}^{i_r}, \underline{e}_{j_1}, \cdots, \underline{e}_{j_s}) \underline{e}_{i_1} \otimes \underline{e}_{i_r} \otimes \overset{\wedge}{\underline{e}}^{j_1} \otimes \overset{\wedge}{\underline{e}}^{j_s}.$$

□

Comment(s): 6.1.3 The numbers $t^{i_1 \cdots i_r}_{j_1 \cdots j_s} = t(\overset{\wedge}{\underline{e}}^{i_1} \cdots, \overset{\wedge}{\underline{e}}^{i_r} \cdots \underline{e}_{j_1} \otimes \underline{e}_{j_s})$ are called the components of the tensor t with respect to the basis $\underline{e}_j\}$. Apparently these numbers are unambiguously.

Example(s): 6.1.1 The Kronecker-delta is the tensor $\delta \in T^1_1(E)$ which belongs to the identical transformation $I \in L(E; E)$ under the canonical isomorphism $T^1_1(E) \simeq L(E; E)$. I.e. $\forall \underline{x} \in E \, \forall \overset{\wedge}{\underline{p}} \in E^* \quad \delta(\overset{\wedge}{\underline{p}}, \underline{x}) = (\overset{\wedge}{\underline{p}}, \underline{x})$.
The components are δ^i_j with respect to every base.

Example(s): 6.1.2 The inner product on E can be interpreted as a map $i(\underline{x}) : T^1_1(E) \rightarrow T^1_0(E)$ in $(i(\underline{x})\, t)(\overset{\wedge}{\underline{p}}) = t(\overset{\wedge}{\underline{p}}, \underline{x})$.

Finally we discuss how linear transformations of a vector space E to a vector space F can be "expanded" to linear transformations between the vector spaces $T^r_s(E)$ and $T^r_s(F)$.

If $P \in L(E; F)$ then also, pure notational, $P \in L(T^1_0(E), T^1_0(F))$. The "pull-back transformation" or simple "pull-back" $P_* \in L(F^*, E^*) = L(T^0_1(E); T^0_1(F))$ is defined by

$$< P_*(\overset{\wedge}{\underline{f}}), \underline{x} > = < \overset{\wedge}{\underline{f}}, P\underline{x} >$$

with $\overset{\wedge}{\underline{f}} \in F^*$ and $\underline{x} \in E$.
Sometimes it is "unhandy" that P_* develops in the wrong direction, but this can be repaired if P is an isomorphism, so if $P^{-1} : F \rightarrow E$, exists.

Definition 6.1.2 Let $P : E \rightarrow F$ be an isomorphism. Then is $P^r_s : T^r_s(E) \rightarrow T^r_s(F)$ defined by

$$P^r_s(t)(\overset{\wedge}{\underline{q}}_1, \cdots, \overset{\wedge}{\underline{q}}_r, \underline{y}_1, \cdots, \underline{y}_s) = t(P_*\overset{\wedge}{\underline{q}}_1, \cdots, P_*\overset{\wedge}{\underline{q}}_r, P^{-1}\underline{y}_1, \cdots, P^{-1}\underline{y}_s)$$

□

Comment(s): 6.1.4 $P^1_0 = (P^{-1})_*$ is a "push-forward", so it works the same directions as P.

The following theorem says that "lifting up the isomorphism P to tensor spaces" has all the desired properties, you expect. Chic expressed: The addition $P \rightarrow P^s_r$ is a covariant functor.

Theorem 6.1.3 Given are the isomorphisms $P : E \rightarrow F$, $Q : F \rightarrow G$, then holds

i. $(P \circ Q)^r_s = P^r_s \circ Q^r_s$.
ii. If $J : E \rightarrow E$ is the identical transformation then is $J^s_r : T^s_r(E) \rightarrow T^s_r(E)$ also the identical transformation.
iii. $P^r_s : T^s_r(E) \rightarrow T^s_r(E)$ is an isomorphism and $(P^r_s)^{-1} = (P^{-1})^r_s$.

□

Proof The proof is straightforward. □

At the end, for the index-fetishists[6].

Theorem 6.1.4 Given

- Vector space E with base $\{\underline{e}_i\}$.
- Vector space F with base $\{\underline{f}_j\}$.
- Isomorphism $P : E \rightarrow F$.

Notate $P\,\underline{e}_i = P^j_i \underline{b}_j$ and $(P^{-1})_* \hat{\underline{e}}^k = Q^k_l \hat{\underline{f}}^l$. Then holds

- $P^j_i Q^i_k = Q^j_i P^i_k = \delta^j_k$.
- For $t \in T^s_r(E)$ with components $t^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ with respect to $\{\underline{e}_i\}$ holds: The components of $P^r_s t \in T^r_s(F)$, with respect to $\{\underline{f}_j\}$ are given by

$$(P^r_s t)^{k_1 \cdots k_r}_{l_1 \cdots l_s} = P^{k_1}_{i_1} \cdot \cdots \cdot P^{k_r}_{i_r} \cdot Q^{i_1}_{l_1} \cdot \cdots \cdot Q^{j_s}_{l_s} \cdot t^{i_1 \cdots i_r}_{j_1 \cdots j_s}.$$

□

Proof □

Comment(s): 6.1.5 Also holds $P^{-1}\,\underline{f}_j = Q^k_j\,\underline{e}_k$.

Section 6.2 The Stokes Equations in (Orthogonal) Curvilinear Coordinates

6.2.1 Introduction

The Stokes equations play an important rule in the theory of the incompressible viscous Newtonian fluid mechanics. The Stokes equations can be written as one vector-valued second order partial differential equation,

$$\operatorname{grad} p = \eta \triangle \mathbf{u}, \tag{6.1}$$

with p the pressure, U the velocity field and η the dynamic viscosity. The Stokes equations express the freedom of divergence of the stress tensor. This stress tensor, say S, is a $\binom{2}{0}$-tensor field, which can be written as

$$S = -p\, I + \eta \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right). \tag{6.2}$$

The herein occuring $\binom{2}{0}$-tensor field $\nabla \mathbf{u}$ (yet not to confuse with the covariant derivative of $\mathbf{u}$) is called the velocity gradient field. But, what exactly is the gradient of a velocity field, and evenso, what is the divergence of a $\binom{2}{0}$-tensor field? This differentiation operations are quite often not properly handled in the literature. In this appendix we put on the finishing touches.

6.2.2 The Stress Tensor and the Stokes equations in Cartesian Coordinates

We consider 6.1 and 6.2 on a domain $\Omega \subset \mathbb{R}^n$. Let $\{X^i\}$ be the Cartesian coordinates on Ω.

6.2.3 The Stress Tensor and the Stokes equations in Arbitrary Coordinates

6.2.4 The Extended Divergence and Gradient in Orthogonal Curvilinear Coordinates

6.2.4.1 The Extended Gradient

6.2.4.2 The Extended Divergence

Section 6.3 The theory of special relativity according Einstein and Minovski

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Section 6.4 Brief sketch about the general theory of special relativity

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Section 6.5 Lattices and Reciproke Bases. Piezoelectricity.

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Section 6.6 Some tensors out of the continuum mechanics.

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Section 6.7 Thermodynamics and Differential Forms.

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Figures

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