Optimal current patterns using stream function pairs

Introduction

MRI scanners are a crucial tool for diagnosing a wide range of afflictions, most prominently cancer. By magnetizing hydrogen atoms in a body, it becomes possible to visualize the different kinds of tissues that are present.

One important type of electromagnet in an MRI system is the so-called gradient coil. These coils supply a linearly sloped, time varying magnetic field in the three Cartesian degrees of freedom (i.e. x, y, z) by means of a carefully designed electric current pattern. By inducing these gradients on the magnetic field, it becomes possible to distinguish scanned regions from each other.

Previously, Philips has successfully developed a method for determining optimal current patterns of a gradient coil using stream functions [1]. With this technique, it became possible to design gradient coils in a methodical and sophisticated way.

However, a limitation of standard stream functions is that they can only be defined on surfaces. A natural extension would therefore be to employ a pair of stream functions that are defined in a volume [2]. This could potentially open up an entire range of new gradient coil designs.

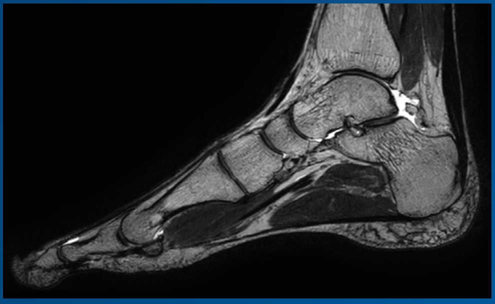

Figure 1: MRI image of a foot

Figure 2: Gradient coil

Mathematical background

Consider the source-free vector field $\boldsymbol{u}$ in $\mathbb{R}^2$. A source-free vector field, assuming it to be continously differentiable, will always be divergence free:

$$\operatorname{div} \boldsymbol{u} := \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

Given this relation, the vector field $\boldsymbol{u}(\boldsymbol{x})$ can be given by a scalar field $\psi(\boldsymbol{x})$. These are related by:

$$u_x = \frac{\partial \psi}{\partial y}, u_y = -\frac{\partial \psi}{\partial x}.$$

Verify this by substituting these expressions into $\operatorname{div} \boldsymbol{u}$. Scalar field $\psi(\boldsymbol{x})$ is called a stream function.

Stream functions can also be used to describe source-free vector fields on a surface S in $\mathbb{R}^3$. Consider a parametric representation of S as a mapping from $U \subset \mathbb{R}^2$ to $\mathbb{R}^3$ in cartesian coordinates: $S = \{\boldsymbol{x} \mid \boldsymbol{x} = \boldsymbol{x}(u, v), (u, v) \in U\}$.

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If the stream function is mapped accordingly by $\psi(u,v) \coloneqq \psi(\boldsymbol{x}(u,v))$ we get:

$$\boldsymbol{u}(u,v) = \frac{\frac{\partial\psi}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial\psi}{\partial u}\frac{\partial x}{\partial v}}{\left\|\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v}\right\|}$$

The motivation for employing stream functions is that we can use a scalar field to describe a (source-free) vector field. This reduces the number of degrees of freedom in for example optimization problems. By illustration, $\boldsymbol{u}(\boldsymbol{x})$ can be identified with the electrical current density on a surface, allowing us to calculate physical quantities and define optimization problems in terms of $\psi(\boldsymbol{x})$.

In case $\boldsymbol{u}(\boldsymbol{x})$ is a divergence-free vector field in $\mathbb{R}^3$, it can be expressed by a pair of stream functions:

$$\boldsymbol{u} \coloneqq \operatorname{grad}\psi \times \operatorname{grad}\eta$$

This holds, since:

$$\operatorname{div}\boldsymbol{u} = \operatorname{grad}\eta \cdot \operatorname{curl}\operatorname{grad}\psi - \operatorname{grad}\psi \cdot \operatorname{curl}\operatorname{grad}\eta = 0$$

However, in order to use this 3D definition, effectively, expressions need to be found for the relevant (physical) quantities. Typically, these are given by integrals of the form:

$$\boldsymbol{h}(\boldsymbol{x}_0) = \iiint_V K(\boldsymbol{x},\boldsymbol{x}_0)\boldsymbol{u}(\boldsymbol{x})dV$$

In order to formulate efficient numerical schemes in terms of stream functions, these integrals should be evaluated (at least in part) analytically for the discretization of choice (probably tetraheders).

For more background information, please refer to [1].

Objectives

- Express physical quantities in terms of stream function pairs.
- Develop an optimization method using these expressions.

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References

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[2] Keller, J.J. (1996) *A pair of stream functions for three-dimensional vortex flows*. Zeitschrift fur Angewandte Mathematik und Physik. 47(6):821–836.