Project Proposals

All project proposals are merely startup suggestions, open for adaptation along the way. Some (but not all) may be narrowed down or scaled up depending on nominal project duration and academic phase (either as a Bachelor Final project or as a Master Final Project). The challenging nature of each project is indicated by a five-star rating along two dimensions: mathematical difficulty and creativity. This rating may be updated depending on actual project execution.

☆☆☆☆☆ trivial

★☆☆☆☆ straightforward (e.g. literature study readily providing comprehensive account)

★★☆☆☆ moderately challenging (e.g. literature study providing key elements)

★★★☆☆ somewhat challenging (e.g. literature study providing strong hints)

★★★★☆ very challenging (e.g. literature study providing minor cues and/or clues)

★★★★★ extremely challenging (unsolved issues, leading edge, innovative thinking)

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Project I. Mathematics: ★★★★☆ — Creativity: ★★☆☆☆

Topic: Hamilton-Jacobi theory

Background and problem formulation. In Riemann-Finsler geometry, geodesics are curves that 'locally' (i.e. among all neighbouring curves in some precise topological sense) minimise a generalised length functional. The general goal in this bachelor final project is to give a concise, mathematically rigorous overview of different characterisations of geodesics, geodesic families and geodesic fields, and their mutual relations. The specific goal is to relate the theory for (parameter invariant) 1-homogeneous Lagrangians (norms) to that of (affinely parametrised) 2-homogeneous Lagrangians (quadratic costs), as well as to relate both Lagrangian formalisms to their Hamiltonian counterparts. The role of the Hamilton-Jacobi equations, in evolutionary and reduced forms for both cases, is of special interest.

Expected output. The goal of this project is to provide a concise tutorial of Hamilton-Jacobi theory for geodesic curves in n dimensions, with a focus on the relation between parameter-variant and parameter-invariant cases.

Societal relevance. Geodesics, in their various manifestations, play an important role in human brain mapping, in which they can be related to white matter tracts (nerve bundles) inferred from diffusion weighted magnetic resonance imaging. As such they are of great importance for understanding brain structure and for medical applications that rely on such insight, such as brain surgery for tumor resection and deep brain stimulation for the treatment of neurological disorders (parkinson, epilepsy, etc.).

Requirements. This project requires familiarity with differential geometry and with ordinary and partial differential equations, all at an introductory bachelor level. The project relies, to a large extent, on a literature survey (calculus of variations, metric geometry, differential equations).

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl).

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Project II. Mathematics: ★★★★☆ — Creativity: ★★★★☆

Topic: Geometry of the Visual System

Background and problem formulation. The conjecture that static images are sections of a 1-dimensional vector bundle over space has been used for image processing purposes, such as inpainting (cf. the Healing Brush in Adobe Photoshop), and may explain various visual illusions. Questions that arise are: How does the visual system disambiguate the gauge field of the linear connection automatically and nearly instantaneously? How can we fix a suitable gauge field for technological applications such as image restoration, and how can we implement this numerically on a computer? (Mimic the Healing Brush.) Can we generalize this to spacetime? For vision, think of explaining the motion-after effect. For image processing, think of video restoration.

Expected output. The goal of this project is to gain insight in human vision and exploit this insight in image processing applications.

Societal relevance. Imaging is one of the main technologies for collecting empirical data involving physical fields of various natures. Fundamental insight is needed to exploit such data. The visual system is our main source of inspiration. Insight in the inner workings of vision may be turned into algorithms for image processing.

Requirements. This project requires familiarity with differential geometry (notably the concept of fiber bundles) and, at an introductory bachelor level, with elliptical partial differential equations. The project relies, to a large extent, on a literature survey (calculus of variations, differential geometry, differential equations, topical literature).

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl).

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Project III. Mathematics: ★★★★★ — Creativity: ★★★★☆

Topic: Riemann-Finsler Geometry for Diffusion Weighted Imaging

Background and problem formulation. In diffusion weighted MRI (DWI) one obtains an attenuated signal on the slit cotangent bundle in 3-space, in which the attenuation is caused by hydrogen spin diffusion mediated by mobile water. Questions that arise are: How does the DWI signal relate to the mean apparent propagator, the locally averaged probability density for spatial transitions of hydrogen spin over a given diffusion time. How do these relate to a (dual) Finsler function for geodesic tractography and connectivity? How can we operationalize such relationships in terms of a computer algorithm that converts DWI data to a (dual) Finsler function?

Expected output. The goal of this project is to relate hydrogen spin diffusion statistics to a Riemann-Finsler metric. A rigorous model might lead to a journal publication.

Societal relevance. A thorough mathematical underpinning of non-Gaussian diffusion processes and their connection to Riemann-Finsler geometry may be used for designing effective image analysis algorithms for diffusion weighted imaging, a promising but notoriously hard and still clinically underexploited imaging modality for probing water mobility in the brain. A rigorous relation between water diffusivity and geometry would be a major stepping stone towards unravelling the human brain connectome.

Requirements. This project requires familiarity with Riemann-Finsler geometry and (non-Gaussian) diffusion processes in porous media.

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl).

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Project IV. Mathematics: ★★★☆☆ — Creativity: ★★★★☆

Topic: Time-Causal Real-Time Systems

Background and problem formulation. In a real-time signal processing system (such as the visual system) the present moment obviously poses a fundamental time horizon beyond which no visual input data are accessible. Moreover, any real-time system processes data with a system-specific finite delay before any conclusions can be drawn based on the empirical evidence gathered. This delay confronts us with an operational time horizon which lies strictly in the past relative to the present moment. This affects our time perception, a phenomenon studied for more than a century in psychology and neuroscience, giving rise to such notions as the 'specious present' (the time interval preceding the physical present perceived as 'now'). The history of data collected over the past prior to this operational time horizon posed by the system's delay may serve to anticipate the 'immediate past', i.e. the operationally inaccessible events that have occurred less than a unit delay prior to the physical present. Questions that arise are: What is the appropriate time parametrisation for a causal real-time system? What are the generic tools for real-time signal processing? Under which mathematical conditions can we 'predict the past' in a profitable way? How can we implement such tools in terms of highly efficient computer algorithms?

Expected output. The precise form of output will depend on creative turns taken during execution and on progress. The pivotal target is to find an operational answer, in mathematical terms, to the question: Can we manipulate the specious moment to our advantage?

Societal relevance. Real-time systems are omnipresent. Minimising delays in the action-reaction cycle is often crucial. Being able to predict this 'immediate past' would allow one to anticipate events already before these become available for analysis and consecutive action. Think of high-frequency trading ('flitshandel') in the equity market. Creative mathematical methods are the precursors of new algorithms and applications to achieve this goal.

Requirements. This project requires familiarity with standard analysis and with the basics of 'early vision', for which ample literature exists in the form of a few popular books on human vision as well as more focused, in-depth scientific literature addressing temporal visual processing and causality aspects.

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl).

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Project V. Mathematics: ★★★★☆ — Creativity: ★★☆☆☆

Topic: Noether's Theorem

Background and problem formulation. The impact of Emmy Noether on physics and mathematics can hardly be overestimated. Her achievements are especially remarkable if you take into account the disheartening circumstances of female talents at the beginning of the 20th century. Even in her prime she was not able to secure a teaching role at her university, despite being championed by eminent male scientists such as Hilbert and Einstein. If the Nobel Prize had been a posthumous award, Emmy Noether would surely be a laureate. One of her contributions, nowadays known as 'Noether's theorem', has been indispensable for a self-consistent formulation of Einstein's general relativity theory, but it reaches far beyond that. It establishes a fundamental connection between symmetries (or invariances) and conservation laws for dynamical systems.

Expected output. The goal of this project is a tribute to Emmy Noether's work in the form of a literature study on Noether's theorem and its impact in general relativity theory or other research areas. The expected output is a concise, mathematically rigorous tutorial account of Noether's theorem in all generality, covering a general class of domains (space, time, spacetime) and codomains (scalars, vectors, tensors).

Societal relevance. Noether's theorem is one of the cornerstones of modern theoretical physics (notably dynamical systems and field theories), since symmetry is often a key element.

Requirements. This project requires familiarity with differential geometry and calculus of variations. Familiarity with Einstein's general relativity theory is useful (and especially recommended for physics students), but not necessary. The books by Dwight Neuenschwander, 'Emmy Noether's Wonderful Theorem', and by David Lovelock and Hanno Rund, 'Tensors, Differential Forms, and Variational Principles' are highly recommended sources for literature study.

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl).

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Project VI. Mathematics: ★★★★★ — Creativity: ★★★☆☆

Topic: Path Integrals

Background and problem formulation. In a path (or functional) integral, the integrand is a functional, i.e. a scalar function defined on a space of functions. Path integrals arise in quantum physics (introduced by Feynman), in probability theory, and in many other fields. They are, however, notoriously hard to handle in a mathematically rigorous fashion. Only a rather restricted class of functionals has been given a solid mathematical basis so far. This subclass is, however, a particularly important one, since it forms the basis of many perturbative expansions of more complicated ones, in which small physical parameters allow a Taylor-like series approximation (Feynman diagrams, for example, in the context of quantum field theories).

Expected output. The goal of this project is to pick up on exact results known for a particular subclass of path integrals, and to write a concise, mathematically rigorous tutorial account. A possible extension is to show how such exact results can be used to handle a more general class of path integrals by approximation and/or to illustrate their practical applicability in the context of a concrete problem.

Societal relevance. Path integrals lack a mathematically rigorous basis, despite their frequent use in state of the art field theories. Both established theories exploiting path integrals as well as new applications would benefit from new insights in the mathematical basis, including numerical handling, of functional integration.

Requirements. The project requires familiarity with advanced functional analysis. There is a vast body of literature from the physics community (for which a physics background would be particularly helpful), but the emphasis here is on mathematical aspects ('mathematical physics').

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl).

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Project VII. Mathematics: ★★★★★ — Creativity: ★★★☆☆

Topic: Pseudo-Finsler Geometry

Background and problem formulation. Riemannian geometry is the study of Riemannian manifolds, i.e. smooth manifolds endowed with a Riemannian metric, defined as an inner product on each tangent space, varying smoothly over the manifold. The Riemannian structure allows one to define the length of a curve as a meaningful geometric quantity, i.e. independent of its parameterization, in terms of a norm (square root of a quadratic form) induced by an inner product. One can generalize this in several ways:

1. By allowing (the components of) the metric tensor (the coefficients of the quadratic form) to vary not only with position, but also with direction. This leads us into the realm of Finsler geometry, the most general geometric setting in which one can naturally define the lengths of curves on a manifold. Such a metric is then referred to as a Finsler metric.
2. By allowing the metric tensor to define a (nondegenerate) pseudo inner product, rather than a positive definite one. In particular we may choose it to be of Lorentzian signature. This leads to Lorentzian geometry, which forms the mathematical basis for Einstein's general theory of relativity.

By generalizing along both ways simultaneously one arrives at pseudo-Finsler geometry (the modifier 'pseudo', crucial in this project, is often omitted). This is one of the research themes pursued in the Applied Differential Geometry group.

Expected output. The project will investigate certain mathematical aspects of the theory, and depending on the background and ambitions of the student, might also investigate, as an application to physics, certain aspects of Finslerian modifications of Einstein's general relativity.

Requirements. The student should have a background in differential geometry, for instance at the level of the course Tensor Calculus and Differential Geometry (2WAH0) by Luc Florack.

Information. If you are interested in this project, please contact Andrea Fuster (A.Fuster@tue.nl) or Sjors Heefer (S.J.Heefer@tue.nl). Useful references: Bao, Chern & Chen, 'An Introduction to Riemann-Finsler Geometry'; arXiv:1804:09727; arXiv: 2003.02300.

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Project VIII. Mathematics: ★★★★☆ — Creativity: ★★★★☆

Topic: Neuroimaging for Neurosurgery

Background and problem formulation. In an ongoing project with the Neurosurgery Department of Elisabeth-TweeSteden Ziekenhuis (ETZ) in Tilburg we apply so-called tractography to find major neural pathways (a.k.a. fiber tracts) in the brain. The premise is that one can find such pathways as shortest paths relative to a metric derived from scanner data using diffusion weighted magnetic resonance imaging. These data reveal the anisotropic nature of water diffusion in the brain, reflecting preferred orientations induced by the fibrous structure of the brain's white matter (bundles of elongated axons connecting nerve cells in surrounding grey matter regions). Tractography pertains to the inverse problem of retrieving this fibrous structure from the observed anisotropies.

In the case of patients suffering from a brain tumor, data in the vicinity of the tumor are affected by edema (swelling due to excessive accumulation of water), which lowers the anisotropy relative to normal white matter regions, thereby complicating tractography algorithms.

The imaging modality of interest is diffusion weighted magnetic resonance imaging, notably a model known as diffusion tensor imaging (DTI). DTI provides a symmetric positive-definite matrix field, which can be visualized as a distribution of ellipsoidal figures by considering the unit level set of the associated quadratic form at each point in the brain. The anisotropy of these figures betrays a preferred orientation induced by anisotropic diffusion of water in the brain (a porous medium in which axon bundles create free diffusion channels, roughly speaking), and provides the basic cue for any tractography algorithm.

Expected output. In this project we aim to 'correct' the data for the swelling effect caused by edema. The idea is to use a geometric method for so-called 'in-painting' known from image processing. This method, in turn, is inspired by quantum field theories studied in theoretical physics, in which a similar geometric device is used to account for local phase invariance of complex fields.

Requirements. This project requires affinity with differential geometry and numerical methods for partial differential equations at an introductory level, as well as some programming skills in Mathematica.

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl) or Rick Sengers (H.J.C.E.Sengers@tue.nl).

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Project IX. Mathematics: ★★★★☆ — Creativity: ★★★☆☆

Topic: Tractometry for Connectomics

Background and problem formulation. In an ongoing project with the Neurosurgery Department of Elisabeth-TweeSteden Ziekenhuis in Tilburg we investigate the feasibility of so-called geodesic tractography to find major neural pathways (a.k.a. fiber tracts) in the brain. The premise is that one can find such pathways as shortest paths relative to a Riemannian (or, more generally, Finslerian) metric derived from diffusion weighted magnetic resonance imaging. A feasibility study indicates that this is indeed the case, albeit at the expense of many 'false positives', i.e. spurious tracts that need to be pruned in order to retain only the biologically most plausible ones. False positives are a necessary consequence of the Hopf-Rinow theorem, or geodesic completeness. In order to remove them we want to investigate connectivity criteria based on a systematic study of (algebraic, differential, integral) invariants that can be defined for each tentative tract. This is what the term *tractometry* refers to.

Expected output. The goal is to define operational criteria in terms of tractometric invariants that effectively remove false positive tracts while retaining true positive ones. Simulation data together with ground truth and experimentally obtained tracts are available. Clinical data may be made available under privacy restrictions.

Requirements. This project requires a strong affinity with differential geometry, and mathematics in general, as well as programming skills in Mathematica. The course Tensor Calculus & Differential Geometry is highly recommended.

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl), Rick Sengers (H.J.C.E.Sengers@tue.nl), or Andrea Fuster (A.Fuster@tue.nl).

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Project X. Mathematics: ★★★★☆ — Creativity: ★★★☆☆

Topic: Ricci Flow for Complete Non-Compact Manifolds with Bounded Curvature

Background and problem formulation. Ricci flow is an evolution of a Riemannian metric producing a one-parameter family of progressively coarsened geometries. The topological nature of the (initial) Riemannian manifold plays an important role in this process.

Societal relevance. Riemannian manifolds are often encountered in the description of physical phenomena as well as in engineering systems. In these contexts the finite resolution of the underlying Riemannian metric is often ignored, which implies that there are no a priori bounds on magnitude and graininess of curvature. An artificial smoothness constraint alleviates mathematical complications to some degree, but lacks physical significance. Ricci flow is a way to embed a Riemannian metric of arbitrary resolution into a family of progressively coarsened geometries. Its effect can be qualitatively compared to diffusion of a scalar field on a flat manifold via the heat equation, in which case the evolution parameter serves as an inverse resolution parameter. The evolution parameter provides a control parameter for the complexity of a Riemannian metric on a finite neighbourhood, which creates opportunities for various coarse-to-fine geometric algorithms.

Expected Output. In this project we are interested in answering questions of existence and uniqueness and in building numerical operationalizations of Ricci flow initialized by a Riemannian metric of bounded curvature on a complete non-compact manifold.

Requirements. This project requires affinity with differential geometry and numerical methods for partial differential equations at an introductory level, as well as some programming skills in Mathematica and/or Python.

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl).

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Project XI. Mathematics: ★★★★☆ — Creativity: ★★★☆☆

Topic: The Breakdown of Geodesic Completeness towards Metric Singularities

Background and problem formulation. In the field of tractography, which seeks to reconstruct neural pathways in the brain from diffusion weighted magnetic resonance imaging, there is a plethora of models and algorithms for tract reconstruction. Mathematically speaking, one can distinguish two mainstream paradigms. The first is based on the integral curves, or streamlines, of a system of first order o.d.e.'s. The second is based on families of curves determined by a system of second order o.d.e.'s.

In the first case a single seed point suffices to disambiguate a tentative tract. Odds are that if you pick a *pair* of points in generic position, no streamline connecting these will exist. In the second case one needs two side conditions, either two initial conditions or two boundary conditions.

In the case of geodesic tractography, a second order scheme, one considers so-called geodesics in a geodesically complete Riemannian (or Finslerian) space. The Hopf-Rinow theorem guarantees the existence of at least one integral curve between any pair of points.

Both paradigms have their pros and cons. In first order schemes one faces the problem of 'false negatives', i.e. missing connections between two points that are (somehow) known to be connected. The second order scheme confronts us with infinitely many 'false positives', which need to be pruned using additional mathematical tools, such as tractrometric characterization and classification. The geodesic rationale allows us to connect both paradigms by considering a family of regular Riemannian metrics with a singular limit. In this project we wish to investigate the question how exactly this family might reconcile the two paradigms, and in particular, how the admissibility of two boundary conditions (i.e. geodesic completeness) gradually 'breaks down' towards the singular limit. For instance, one may study the effect of path length of a fiducial geodesic as one approaches the singular limit, given a pair of end points.

Societal relevance. Removing false positives and avoiding false negatives is a crucial requirement in clinical validation of tractography for neurosurgical and neurological applications. Reconciliation of the two prevalent tractography paradigms will allow us to correctly interpret their qualitative difference.

Expected Output. We wish to understand the unfolding of geodesic tracts under a singularity approaching family of Riemannian metrics in terms of the evolution of tractometric features, such as their path length, given arbitrary end point conditions.

Requirements. This project requires affinity with differential geometry and numerical methods for ordinary differential equations at an introductory level, as well as some programming skills in Mathematica and/or Python.

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl).

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Project XII. Mathematics: ★★★★★ — Creativity: ★★★☆☆

Topic: Lagrangian and Hamiltonian Formulations of General Relativity Theory Based on the Einstein-Hilbert Action

Background and problem formulation. Einstein's theory of general relativity is typically formulated using a covariant Lagrangian framework, in which the Einstein field equations are obtained via a variational principle from the so-called Einstein-Hilbert action (with appropriate boundary terms). For simplicity we will refer to this approach as the *covariant formalism*. However, the same theory can also be formulated using a Hamiltonian framework, referred to as the *canonical formalism*. In this framework one distinguishes metric components and their conjugate momenta as a priori independent phase space variables, linked via Hamilton's equations, the first order evolutionary counterpart of the second order covariant field equations. The canonical formalism is of interest in certain attempts to formulate a quantum theory of gravity.

Expected output. The goal of this project is to gain insight in the variational or action principle from both Lagrangian and Hamiltonian viewpoints, geared towards its use in formulating Einstein's general relativity theory from both perspectives.

Societal relevance. This project is curiosity driven and its relevance is mostly scientific. Deep insight in Einstein's theory is also of practical relevance for conducting experimental research of gravitational phenomena, such as gravitational waves.

Requirements. This project requires familiarity with aspects of both mathematics and physics, notably differential geometry, classical mechanics, relativity theory, ordinary and partial differential equations, variational principle, et cetera, all at an introductory bachelor level. The project relies, to a large extent, on a literature survey, but due to the relevance of both physics and mathematics is most suited for a student combining both curricula.

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl).

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Project XIII. Mathematical Physics: ★★★★★ — Creativity: ★★★☆☆

Topic: Quantum-Mechanical Basis of the Bloch-Torrey Equations

Background and problem formulation. The so-called Bloch equations describe the time evolution of the nuclear magnetization of a macroscopic system of nuclear magnetic spins in a magnetic field. The Bloch-Torrey equations are a modification of these that take into account diffusion of nuclear spins. They are fundamental in understanding processes like nuclear magnetic resonance (NMR) and magnetic resonance imaging (MRI, respectively DWI). At the microscopic level such processes are governed by the time-dependent Schrödinger equation, or rather the Pauli equation, applicable to spin-half fermions like the proton.

Expected output. The goal of this project is to couple the two regimes by rigorously deriving the Bloch-Torrey equations from the underlying Pauli equation. This involves simplifying assumptions and approximations (rotating wave approximation with slow oscillations near Larmor frequency, Born's weak coupling approximation, Markov's fast relaxation approximation, et cetera). The expected output is a clarification of all mathematical steps involved, including explicit assumptions and their physical justifications.

Societal relevance. This project is curiosity driven and its relevance is mostly scientific. The Bloch-Torrey equations are fundamental to understanding DWI, the only non-invasive in-vivo modality for probing the neural structure of the human brain (limited by MRI resolution). As such they form the mathematical basis for the 'Holy Grail' of neuroimaging, the unravelling of the human structural connectome.

Requirements. This project requires familiarity with aspects of both mathematics and physics, notably quantum mechanics, ordinary and partial differential equations, et cetera. The project relies, to a large extent, on a literature survey, but due to the relevance of both physics and mathematics is most suited for a student combining both curricula.

Information. If you are interested in this project, please contact Luc Florack (L.M.J.Florack@tue.nl).

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Project XIV.Mathematics: ★★★★★ — Creativity: ★★★☆☆

Topic: Local vs global metrizability of linear connections on the tangent bundle of a smooth manifold

Background and problem formulation. A linear connection on the tangent bundle of a smooth manifold M is a mathematical object that determines a collection of trajectories on M, called autoparallel curves—the straightest possible curves allowed by the curvature of the connection (think e.g. of great circles on a sphere). Another type of mathematical object that can be defined on M is a (pseudo-)Riemannian metric, which gives rise to the notion of distances and angles. A classical result by Levi-Civita states that any (pseudo-)Riemannian metric uniquely determines a linear connection on the tangent bundle—the Levi-Civita connection, whose autoparallel curves are precisely the geodesics (i.e. length-extremizing curves) of the metric. However, not any linear connection is a Levi-Civita connection of some (pseudo-)Riemannian metric. If it is, then we say that the connection is *metrizable*.

Expected output. This project is about this 'inverse problem': given a linear connection, then under what conditions is this connection metrizable? In other words, when can we find a pseudo-Riemannian metric that has the given connection as its Levi-Civita connection? More specifically, one of the aims of this project is to understand the difference between local (i.e. in a neighborhood of some point in M) or global (on all of M at once) metrizability. A possible ultimate goal of the project could be to study and try to understand the situation for a specific type of linear connection, namely one induced (as a kind of generalized Levi-Civita connection) by an m-Kropina metric—a specific type of Finsler metric (Finsler metrics are a generalization of pseudo-Riemannian metrics). But this depends on the available time, circumstances, student's preference etc. Alternatively, a major part of the project could be devoted to a literature study of known results in the field.

Societal relevance. This project is curiosity driven and its relevance is mostly academic.

Requirements. This project requires familiarity with differential geometry and a strong affinity with rigorous and abstract mathematical thinking. Experience with point-set topology would be a plus, but this is not required; this could be learned in due course.

Information. If you are interested in this project, please contact Sjors Heefer (S.J.Heefer@tue.nl). Useful references: https://arxiv.org/abs/2404.09858

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Project XV. Mathematics/mathematical physics: ★★★★☆ — Creativity: ★★★☆☆

Topic: Finsler geometry and (optionally) Finsler gravity: a perturbative approach

Background and problem formulation. In Einstein's general theory of relativity (GR), the gravitational 'force' is not seen as a force but as a manifestation of the geometry of the 4-dimensional spacetime continuum (hereafter simply *spacetime*). The theory rests on the foundation that this geometry can be described by a pseudo-Riemannian metric. Quantum gravity research suggests that this might not be the case, however, and that a more sophisticated geometric framework is necessary instead, namely that of *Finsler geometry*, a powerful generalization of pseudo-Riemannian geometry. GR can be generalized to incorporate the possibility of spacetime having this kind of geometry, and the resulting theory is called *Finsler gravity.* Since GR describes the world around us very accurately, however, it is natural to assume that the possible (Finslerian) deviation from GR is small, hence the geometry of spacetime should be approximately pseudo-Riemannian.

Expected output. In this project, we consider Finsler geometries that are 'almost' pseudo-Riemannian, i.e. we assume that the metric is given by some pseudo-Riemannian metric plus a small perturbation term. This project aims to study the properties of such a metric. In particular, it would be interesting to study its geodesics, its connection, its curvature, its symmetries (via Killling's equation), etc. Also, we would like to figure out what conditions the perturbation term needs to satisfy in order for the resulting metric to actually satisfy all axioms of a Finsler metric. Depending on the student's background and preference, it is also possible to investigate the physical implications of modelling the geometry of spacetime by such a metric in the context of Finsler gravity (we could try to compute the speed of light, for instance, which might not be constant!). But the project could be also be purely mathematical, if so desired.

Societal relevance. This project is curiosity driven and its relevance is mostly academic/scientific. However, applications of Finsler geometry (e.g. relativity, nerve fiber reconstruction in diffusion MRI, blood vessel tracking, etc.) are becoming more and more pervasive.

Requirements. This project mostly requires familiarity with differential geometry. Since computations in Finsler geometry can become pretty involved, however, experience with computer algebra software like Wolfram Mathematica could prove very useful as well.

Information. If you are interested in this project, please contact Sjors Heefer (S.J.Heefer@tue.nl). Useful references: https://arxiv.org/abs/2404.09858

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Project XVI. Mathematical physics: ★★★★☆ — Creativity: ★★★☆☆

Topic: Time and distance in general relativity and beyond

Background and problem formulation. In Einstein's general theory of relativity, the 4-dimensional spacetime continuum (hereafter simply *spacetime*) is modelled by a pseudo-Riemannian manifold of Lorentzian signature. For a given observer, the notion of time is in general only defined at the very location of the observer itself. Using an accurate clock, an observer can measure time intervals between events happening precisely at her location (in spacetime), and mathematically this time is given by the pseudo-Riemannian length of the curve that the observer takes through spacetime between the two events. However, there is no unique way for this observer to assign a time to events happening far away from her.
One way of assigning a time and distance is given by the radar method. By postulating that light always travels at the speed of light (also globally, which is not at all obvious!), an observer can assign a time and a distance to any other location in spacetime. In flat Minkowski spacetime (describing a situation with no gravity) this method can be used to derive the Lorentz transformations, relating the temporal and spatial coordinates of two observers moving relatively to one another.
Another way of assigning a time and distance is by foliating spacetime by so-called surfaces of simultaneity. These are 3D spatial hypersurfaces orthogonal to the observer's time direction, that each represent the universe at a given instant of 'time', and that together make up spacetime as a whole.

Expected output. In this project, we will apply the radar method and the 'surfaces of simultaneity' method to more interesting spacetime geometries, such as:

- the Schwarzschild geometry, which describes the geometry of spacetime around a static, spherically symmetric mass distribution such as a star or a planet;
- the Friedmann-Lemaître-Robertson-Walker (FLRW) geometry, which describes the expansion of the universe at cosmological scales;
- maximally symmetric spacetime geometries (the spacetime analogues of Euclidean geometry, spherical geometry and hyperbolic geometry).

The overarching question in all of this is what would be the most natural way to define 'time' far away from the observer in each of these scenarios.
Furthermore, if time (and the student's interest) permits, we can also look at generalizations of general relativity based on Finsler geometry, an extension of pseudo-Riemannian geometry.

Societal relevance. This project is curiosity driven and its relevance is mostly scientific.

Requirements. This project mostly requires familiarity with differential geometry. Experience with general relativity would be very useful but not strictly required. All physics that is required for the project can be learned in due course.

Information. If you are interested in this project, please contact Sjors Heefer (S.J.Heefer@tue.nl).

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